LECTURE NOTES ON MEASURE THEORY FALL 2020

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1. Preface.

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LATEX ource of these slides is also available on git.

2. Syllabus Overview

- Class website and full syllabus: http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/720-measure
- TA: Lantian Xu $<\!lxu2@andrew.cmu.edu>$
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterm: Fri Oct 9th (90 mins, self proctored, can be taken any time)
- Zoom lectures:
 - $\triangleright\,$ Please enable video. (It helps me pace lectures).
 - $\triangleright\,$ Mute your mic when you're not speaking. Use head phones if possible. Consent to be recorded.
 - $\triangleright\,$ If I get disconnected, check your email for instructions.
- Homework:
 - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
 - $\triangleright~20\%$ penalty if turned in within an hour of the deadline. 100% penalty after that.
 - $\triangleright\,$ Bottom 20% homework is dropped from your grade (personal emergencies, other deadlines, etc.).
 - $\triangleright\,$ Collaboration is encouraged. Homework is not a test ensure you learn from doing the homework.
 - $\triangleright\,$ You must write solutions independently, and can only turn in solutions you fully understand.
- Exams:
 - $\triangleright\,$ Can be taken at any time on the exam day. Open book. Use of internet allowed.
 - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
 - \triangleright Self proctored: Zoom call (invite me). Record yourself, and your screen to the cloud.
 - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.
- Academic Integrity
 - $\triangleright\,$ Zero tolerance for violations (automatic ${\bf R}).$
 - \triangleright Violations include:

- Not writing up solutions independently and/or plagiarizing solutions
- Turning in solutions you do not understand.
- Seeking, receiving or providing assistance during an exam.
- Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
- \triangleright All violations will be reported to the university, and they may impose additional penalties.
- Grading: 40% homework, 20% midterm, 40% final.

3. Sigma Algebras and Measures

- Motivation: Suppose $f_n: [0,1] \to [0,1]$, and $(f_n) \to 0$ pointwise. Prove $\lim_{n \to \infty} \int_0^1 f_n = 0$.
 - $\triangleright\,$ Simple to state using Riemann integrals. Not so easy to prove. (Challenge!)
 - \triangleright Will prove this using Lebesgue integration.
 - Riemann integration: partition the domain (count sequentially)
 - Lebesgue integration: partition the range (stack and sort).
- Goal:
 - $\triangleright\,$ Develop Lebesgue integration.
 - ▷ Need a notion of "measure" (generalization of volume)
 - \triangleright Need " $\sigma\text{-algebras}$ ".
- Why $\sigma\text{-algebras}?$

Theorem 3.1 (Banach Tarski). There exists $n \in \mathbb{N}$, sets $A_1, \ldots, A_n \subseteq B(0,1) \subseteq \mathbb{R}^3$ such that:

- (1) A_1, \ldots, A_n partition B(0, 1).
- (2) There exist isometries R_i such that $R_1(A_1), \ldots, R_n(A_n)$ partition B(0,2).

• How do you explain this?

Definition 3.2 (σ -algebra). Let X be a set. We say $\Sigma \subseteq \mathcal{P}(X)$ is a σ -algebra on X if:

- (1) Nonempty: $\emptyset \in \Sigma$
- (2) Closed under compliments: $A \in \Sigma \implies A^c \in \Sigma$.
- (3) Closed under countable unions: $A_i \in \Sigma \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Remark 3.3. Any σ -algebra is also closed under countable intersections.

Question 3.4. Is $\mathcal{P}(X)$ is a σ -algebra?

Question 3.5. Is $\Sigma \stackrel{\text{\tiny def}}{=} \{\emptyset, X\}$ is a σ -algebra?

Question 3.6. Is $\Sigma = \{A \mid |A| < \infty \text{ or } |A^c| < \infty\}$ a σ -algebra?

Question 3.7. Is $\Sigma = \{A \mid either A \text{ or } A^c \text{ is finite or countable}\}\ a \sigma$ -algebra?

Proposition 3.8. If $\forall \alpha \in \mathcal{A}$, Σ_{α} is a σ -algebra, then so is $\bigcap_{\alpha \in \mathcal{A}} \Sigma_{\alpha}$.

Definition 3.9. If $\mathcal{E} \subseteq \mathcal{P}(X)$, define $\sigma(\mathcal{E})$ to be the intersection of all σ -algebras containing \mathcal{E} .

Remark 3.10. $\sigma(\mathcal{E})$ is the smallest σ -algebra containing \mathcal{E} .

Definition 3.11. Suppose X is a topological space. The *Borel* σ -algebra on X is defined to be the σ -algebra generated by all open subsets of X. Notation: $\mathcal{B}(X)$.

Question 3.12. Can you get $\mathcal{B}(X)$ by taking all countable unions / intersections of open and closed sets?

Question 3.13. Is $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$?

Definition 3.14. Let Σ be a σ -algebra on X. We say μ is a (positive) measure on (X, Σ) if:

- (1) $\mu: \Sigma \to [0,\infty]$
- (2) $\mu(\emptyset) = 0$
- (3) (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 3.15. Is the second assumption necessary?

Question 3.16. Let $\mu(A) = cardinality of A$. Is μ a measure?

Question 3.17. Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?

Theorem 3.18. There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \operatorname{vol}(I)$ for all cuboids I.

- Goal: Define $\int_X f d\mu$ (the Lebesgue integral).
- Idea:

 \triangleright Say $s: X \to \mathbb{R}$ is such that $s = \sum_{i=1}^{N} a_i \mathbf{1}_{A_i}$, for some $a_i \in \mathbb{R}$, $A_i \in \Sigma$. (Called simple functions.)

- \triangleright Define $\int_X s \, d\mu = \sum_{i=1}^N a_i \mu(A_i).$
- $\triangleright \text{ If } f \ge 0, \text{ define } \int_X f \, d\mu = \sup_{s \leqslant f} \int_X s \, d\mu.$
- Will do this after constructing the Lebesgue measure.
- 4. Lebesgue Measure
- 4.1. Lebesgue Outer Measure.

Definition 4.1. We say $I \subseteq \mathbb{R}$ is a *cell* if I is a finite interval. Define $\ell(I) = \sup I - \inf I$.

Definition 4.2. We say $I \subseteq \mathbb{R}^d$ is a *cell* if it is a product of cells. If $I = I_1 \times \cdots \times I_d$, then define $\ell(I) = \prod_{i=1}^d \ell(I_i)$.

Remark 4.3. $\ell(I) = \ell(\mathring{I}) = \ell(\overline{I}).$

Remark 4.4. $\emptyset = \prod_{1}^{d} (a, a)$, and so $\ell(\emptyset) = 0$.

Remark 4.5. For all $\alpha \in \mathbb{R}^d$, $\ell(I) = \ell(I + \alpha)$.

Theorem 4.6. There exists a (unique) measure λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\lambda(I) = \ell(I)$ for all cells I.

Question 4.7. *How do you extend* ℓ *to other sets?*

Definition 4.8 (Lebesgue outer measure). Given $A \subseteq \mathbb{R}^d$, define $\lambda^*(A) = \inf \left\{ \sum_{1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{1}^{\infty} I_k$, where I_k is a cell $\right\}$.

Remark 4.9. Some authors use m^* instead of λ^* .

Remark 4.10. λ^* is defined on $\mathcal{P}(\mathbb{R}^d)$; but only "well behaved" on a σ -algebra.

Question 4.11. What is $\lambda^*(\emptyset)$? What is $\lambda^*(\mathbb{R}^d)$?

Proposition 4.12. If $E \subseteq F$, then $\lambda^*(E) \leq \lambda^*(F)$.

Proposition 4.13. If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then $\lambda^*(\cup_1^\infty E_i) \leq \sum_{i=1}^\infty \lambda^*(E_i)$.

Proposition 4.14. Let $A, B \subseteq \mathbb{R}^d$, and suppose d(A, B) > 0. Then $\lambda^*(A \cup B) = \lambda * (A) + \lambda * (B)$.

Proof: Only need to show $\lambda^*(A \cup B) \ge \lambda^*(A) + \lambda^*(B)$. If $\lambda^*(A \cup B) = \infty$, we are done, so assume $\lambda^*(A \cup B) < \infty$.

Proposition 4.15. If $I \subseteq \mathbb{R}^d$ is a cell, then $\lambda^*(I) = \ell(I)$.

Lemma 4.16. If $\{I_k\}$ divide I by hyperplanes, then $\sum \ell(I_k) = \ell(I)$.

Lemma 4.17. $\lambda^*(A) = \inf\{\sum \ell(I_i) \mid A \subseteq \bigcup I_k, \text{ and } I_k \text{ are all open cells}\}.$

Proof of Proposition 4.15: Suppose first I is closed (hence compact). Pick $\varepsilon > 0$.

Proposition 4.18 (Translation invariance). For all $A \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $\lambda^*(A) = \lambda^*(\alpha + A)$.

4.2. Carathéodory Extension. Our goal is to start with an outer measure, and restrict it to a measure.

Definition 4.19. We say μ^* is an outer measure on X if:

- (1) $\mu^* \colon \mathcal{P}(X) \to [0, \infty]$, and $\mu^*(\emptyset) = 0$.
- (2) If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.
- (3) If $A_i \subseteq X$ (not necessarily disjoint), then $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Example 4.20. Any measure is an outer measure.

Example 4.21. The Lebesgue outer measure is an outer measure.

Theorem 4.22 (Carathéodory extension). Let $\Sigma \stackrel{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \forall A \subseteq X\}$. Then Σ is a σ -algebra, and μ^* is a measure on (X, Σ) .

Remark 4.23. Clearly $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \ge \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c . If E is nice, hopefully the overlap is small.

Proof of Theorem 4.22

- (1) $\emptyset \in \Sigma$.
- (2) $E \in \Sigma \implies E^c \in \Sigma$.
- (3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \ldots, E_n \in \Sigma \implies \bigcup_{i=1}^{n} E_i \in \Sigma$.)

- (4) If $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\mu^*(A \cap (\bigcup_{i=1}^n E_i)) = \sum_{i=1}^n \mu^*(A \cap E_i)$.
- (5) Σ is closed under countable *disjoint* unions, and μ^* is countably additive on Σ .
 - *Proof:* Let $E_1, E_2, \ldots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

Remark 4.24. Note, the above shows $\mu^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$.

Definition 4.25. Define the Lebesgue σ -algebra by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \cap \lambda^*(A \cap E^c) \ \forall A \subseteq \mathbb{R}^d\}.$

Definition 4.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

Remark 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ -algebra, and λ is a measure on \mathcal{L} .

Question 4.28. Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?

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Proposition 4.29. If I \subseteq \mathbb{R}^d is a cell, then I \in \mathcal{L}(\mathbb{R}^d).
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Proof:

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Proposition 4.30. \mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d).
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Remark 4.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Here are two results that will be proved later:

Theorem 4.32. $\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d)$. (In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.) **Theorem 4.33.** $\mathcal{L}(\mathbb{R}^d) \subseteq \mathcal{P}(\mathbb{R}^d)$.

Theorem 4.34 (Uniqueness). If μ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.

Question 4.35. Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose μ, ν are two measures which agree on \mathcal{E} . Must they agree on $\sigma(E)$?

5. Abstract measures

5.1. Dynkin systems.

Question 5.1. Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?

 $\triangleright\,$ Clearly need Π to be closed under intersections.

Question 5.2. Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ -algebra?

 $\triangleright \text{ If } A, B \in \Lambda, \text{ must } A \cup B \in \Lambda?$ $\triangleright \text{ If } A \subseteq B, A, B \in \Lambda, \text{ must } B - A \in \Lambda?$ $\triangleright \text{ If } A_i \subseteq A_{i+1} \in \Lambda, \text{ must } \cup_1^{\infty} A_i \in \Lambda?$

Definition 5.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ -system if:

(1) $X \in \Lambda$ (2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$. (3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^{\infty} A_n \in \Lambda$.

Definition 5.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π -system if whenever $A, B \in \Pi$, we have $A \cap B \in \Pi$.

Lemma 5.5 (Dynkin system lemma). If Π is a π -system, and $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \sigma(\Pi)$.

Corollary 5.6. If μ , ν are finite measures such that $\mu = \nu$ on Π , and Π is closed under intersections, then $\mu = \nu$ on $\sigma(\Pi)$.

Proof of Lemma 5.5

(1) The arbitrary intersection of λ -systems is a λ -system. So it make sense to talk about $\lambda(\Pi)$. (2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.

- (3) If Λ is both a π -system and a λ -system, then Λ is a σ -algebra.
- (4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.
- (5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. Then Λ_C is a λ -system.
- (6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.
 - \triangleright Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.
 - \triangleright For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

5.2. Regularity of measures.

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

- (1) For all compact sets K, we have $\mu(K) < \infty$.
- (2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}.$
- (3) For all Borel sets A we have $\mu(A) = \inf{\{\mu(U) \mid U \supseteq A, U \text{ open}\}}$.

Motivation:

- \triangleright Approximation of measurable functions by continuous functions
- \triangleright Differentiation of measures
- $\triangleright~$ Uniqueness in the Riesz representation theorem

Question 5.8. If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}$ for all Borel sets A?

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is regular, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}.$

(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\mu(U - C) < \varepsilon$.

(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem.

Theorem 5.10. Suppose X is a compact metric space, and μ is a finite Borel measure on X. Then μ is regular. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ closed such that $\mu(U - K) < \varepsilon$.

Proof:

- (1) Let $\Lambda = \{A \in \mathcal{B}(X) \mid \forall \varepsilon > 0, \exists K \subseteq A \text{ compact}, U \supseteq A \text{ open, such that } \mu(U K) < \varepsilon \}.$
- (2) Λ contains all open sets.
- (3) Λ is a λ -system. (In this case it's easy to directly show that Λ is a σ -algebra.)
- (4) Dynkin's Lemma implies $\Lambda \supseteq \mathcal{B}(X)$, finishing the proof.

Corollary 5.11. Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subset X$ such that $\bar{B}_n \subset \mathring{B}_{n+1}$, \bar{B}_n is compact, $X = \bigcup_{1}^{\infty} B_n$ and $\mu(B_n) < \infty$. Then μ is regular. Further:

 \square

- (1) For any Borel set A, $\mu(A) = \sup\{\mu(K) \mid K \subseteq K \text{ is compact}\}.$
- (2) For any $\varepsilon > 0$, there exists $U \supseteq A$ open and $C \subseteq A$ closed such that $\mu(U C) < \varepsilon$.

Proof. On homework.

Theorem 5.12. Let $A \in \mathcal{L}(\mathbb{R}^d)$, $\lambda(A)$.

- (1) $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A, U \text{ open}\} = \sup\{\lambda(K) \mid K \subseteq A, K \text{ compact}\}.$
- (2) There exists $\varepsilon > 0$, $C \subseteq A$ closed and $U \supseteq A$ open such that $\lambda(U C) < \varepsilon$.

5.3. Non-measurable sets.

Theorem 5.13. There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{L}(R)$.

Proof:

(1) Let $C_{\alpha} = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of \mathbb{R}/\mathbb{Q} containing α .)

(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_{\alpha}| = 1$ for all α . (3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$. (4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$. (5) $\lambda(E) > 0$ (6) $\lambda(E) = 0$ (contradiction).

Theorem 5.14. Let $A \subseteq \mathbb{R}^d$. Every subset of A is Lebesgue measurable if and only if $\lambda(A^*) = 0$.

Proof. One direction is immediate. The other direction is accessible with what we know so far, but we won't do the proof in the interest of time. \Box

5.4. Completion of measures.

Theorem 5.15. $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if there exist $F, G \in \mathcal{B}(\mathbb{R}^d)$ such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$. **Corollary 5.16.** Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$. **Definition 5.17.** Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by $\Sigma_{\mu} \stackrel{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$ For every $A \in \Sigma_{\mu}$, find F, G as above and define $\overline{\mu}(A) = \mu(F)$. **Definition 5.18.** Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma, E \supseteq A, \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Theorem 5.19. Σ_{μ} is a σ -algebra, $\bar{\mu}$ is a measure on Σ_{μ} , and $(X, \Sigma_{\mu}, \bar{\mu})$ is complete.

Theorem 5.20. Σ_{μ} is the smallest μ -complete σ -algebra containing Σ .

Corollary 5.21. $\Sigma_{\mu} = \sigma(\Sigma \cup \mathcal{N}).$

Corollary 5.22. $\mathcal{L}(\mathbb{R}^d) = \sigma(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N}).$

Remark 5.23. There could exist μ -null sets that are not in Σ .

6. Measurable Functions

6.1. Measurable functions.

Definition 6.1. Let (X, Σ, μ) be a measurable space, and (Y, τ) a topological space. We say $f: X \to Y$ is measurable if $f^{-1}(\tau) \subseteq \Sigma$. *Remark* 6.2. Y is typically $[-\infty, \infty]$, \mathbb{R}^d , or some linear space.

Remark 6.3. Any continuous function is Borel measurable, but not conversely.

Question 6.4. Say $f: X \to Y$ is measurable. For every $B \in \mathcal{B}(Y)$, must $f^{-1}(B) \in \Sigma$?

Theorem 6.5. Say $f: X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.

Lemma 6.6. Let $f: X \to Y$ be arbitrary, and Σ be a σ -algebra on X. Then $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$ is a σ -algebra (on Y).

Corollary 6.7. Let $f: X \to [-\infty, \infty]$. Then f is measurable if and only if for all $a \in \mathbb{R}$, we have $\{f < a\} \in \Sigma$.

Lemma 6.8. If $f: X \to \mathbb{R}^m$ is measurable, and $g: \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f: X \to \mathbb{R}^n$ is measurable.

Question 6.9. Is the above true if g was Lebesgue measurable?

Theorem 6.10. Let $f_n: X \to \mathbb{R}$ be a sequence of measurable functions. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, inn f_n, inn f_

Theorem 6.11. Let $f, g: X \to \mathbb{R}$. The function $(f, g): X \to \mathbb{R}^2$ is measurable if and only if both f and g are measurable. Corollary 6.12. If $f, g: X \to \mathbb{R}$ are measurable, then so is f + g, fg and f/g (when defined).

6.2. Cantor Function.

Definition 6.13 (Cantor function). Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_{\alpha}(C \cap [0, x])/H_{\alpha}(C)$.

(1) f(0) = 0, f(1) = 1 and f is increasing. (In fact, f is differentiable exactly on C, and f' = 0 wherever defined.) (2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.) (3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous f(g(x)) = x)).

Proposition 6.14. The function $g: [0,1] \to C$ is a strictly injective Borel measurable function.

Theorem 6.15. $\mathcal{L}(\mathbb{R}) \supseteq \mathcal{B}(\mathbb{R})$.

Theorem 6.16. There exists $h_1, h_2 \colon \mathbb{R} \to \mathbb{R}$ such that h_1 is $\mathcal{L}(\mathbb{R})$ -measurable, h_2 is $\mathcal{B}(\mathbb{R})$ measurable, but $h_1 \circ h_2$ is not $\mathcal{L}(\mathbb{R})$ measurable.

Remark 6.17. Note $h_2 \circ h_1$ has to be $\mathcal{B}(\mathbb{R})$ -measurable.

6.3. Almost Everywhere.

Definition 6.18. Let (X, Σ, μ) be a measure space. We say a property P holds almost everywhere if there exists a null set N such that P holds on N^c .

Example 6.19. If f, g are two functions, we say f = g almost everywhere if $\{f \neq g\}$ is a null set.

Example 6.20. Almost every real number is irrational.

Example 6.21. If $A \in \mathcal{L}(\mathbb{R})$, then $\lim_{h \to 0} \frac{\lambda(A \cap (x, x + h))}{h} = \mathbf{1}_A(x)$ for almost every x. (Contrast with HW3, Q3b)

Example 6.22. Let $x \in (0,1)$, and p_n/q_n be the n^{th} convergent in the continued fraction expansion of x. Then $\lim_{n \to \infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$.

Assume hereafter (X, Σ, μ) is complete.

Proposition 6.23. If f = g almost everywhere and f is measurable, then so is g.

Proposition 6.24. If $(f_n) \to f$ almost everywhere, and each f_n is measurable, then so is f.

6.4. Approximation.

Definition 6.25. A function $s: X \to \mathbb{R}$ is called *simple* if s is measurable, and has finite range (i.e. $s(\mathbb{R}) = \{a_1, \ldots, a_n\}$).

Question 6.26. Why bother with simple functions?

Theorem 6.27. If $f \ge 0$ is a measurable function, then there exists a sequence of simple functions (s_n) which increases to f.

Corollary 6.28. If $f: X \to \mathbb{R}$ is measurable, then there exists a sequence of simple functions (s_n) such that $(s_n) \to f$ pointwise, and $|s_n| \leq |f|$.

Theorem 6.29 (Lusin). Let μ be a finite regular measure on a metric space X. Let $f: X \to \mathbb{R}$ be measurable. For any $\varepsilon > 0$ there exists a continuous function $g: X \to \mathbb{R}$ such that $\mu\{f \neq g\} < \varepsilon$.

Lemma 6.30 (Tietze's extension theorem). If $C \subseteq X$ is continuous, and $f: C \to \mathbb{R}$ is continuous, then there exist $\overline{f}: X \to \mathbb{R}$ such that $\overline{f} = f$ on C.

Lemma 6.31. Let $f: X \to \mathbb{R}$ be measurable. For every $\varepsilon > 0$, there exists $C \subseteq X$ closed such that $\mu(X - C) < \varepsilon$ and $f: C \to \mathbb{R}$ is continuous.

Proof of Lusin's theorem. Previous two lemmas.

Proof of Lemma 6.31.

7. Integration

7.1. Construction of the Lebesgue integral. Recall, $s: X \to \mathbb{R}$ is simple if s is measurable and has finite range.

Definition 7.1. Let $s \ge 0$ be a simple function. Let $\{a_1, \ldots, a_n\} = s(X)$, and set $A_i = s^{-1}(a_i)$. Define $\int_X s \, d\mu = \sum_{i=1}^n a_i A_i$. Remark 7.2. Always use the convention $0 \cdot \infty = 0$.

Remark 7.3. Other notation: $\int_X s \, d\mu = \int_X s(x) \, d\mu(x)$.

Proposition 7.4. If $0 \leq s \leq t$ are simple, then $\int_X s \, d\mu \leq \int_X t \, d\mu$.

Proposition 7.5. If $s, t \ge 0$ are simple, then $\int_X (s+t) d\mu = \int_X s d\mu + \int_X t d\mu$.

Definition 7.6. Let $f: X \to [0, \infty]$ be measurable. Define $\int_X f d\mu = \sup\{\int_X s d\mu \mid 0 \leq s \leq f, s \text{ simple.}\}$.

Definition 7.7. Let $f: X \to [-\infty, \infty]$ be measurable. We say f is *integrable* if $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$. In this case we define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$.

Definition 7.8. We let $L^1(X) = L^1(X, \Sigma, \mu)$ be the set of all integrable functions on X. (Note $f \in L^1 \iff |f| \in L^1$.)

Definition 7.9. We say f is integrable in the extended sense if either $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$. In this case we still define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$.

Remark 7.10. If both $\int_X f^+ d\mu = \infty$ and $\int_X f^- d\mu = \infty$, then $\int_X f d\mu$ is not defined.

Question 7.11. Do we have linearity?

Proposition 7.12 (Consistency). If $s = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \ge 0$ is simple, then $\sum a_i \mu(A_i) = \sup\{\int_X t \, d\mu \mid 0 \le t \le s, \text{ simple}\}.$

Theorem 7.13 (Monotone convergence). Say $(f_n) \to f$ almost everywhere, $0 \leq f_n \leq f_{n+1}$, then $(\int_X f_n d\mu) \to \int_X f d\mu$.

Theorem 7.14. If f, g are integrable, then $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$.

7.2. Dominated convergence. When does $\lim \int_X f_n d\mu \neq \int_X f d\mu$? Two typical situations where it fails:

(1) Mass escapes to infinity

(2) Mass clusters at a point

Theorem 7.15 (Dominated convergence). Say (f_n) is a sequence of measurable functions, such that $(f_n) \to f$ almost everywhere. Moreover, there exists $F \in L^1(X)$ such that $|f_n| \leq F$ almost everywhere. Then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Lemma 7.16 (Fatou). Suppose $f_n \ge 0$, and $(f_n) \to f$. Then $\liminf \int_X f_n \, d\mu \ge \int_X f \, d\mu$.

Proof of Theorem 7.15

Theorem 7.17 (Beppo-Levi). If $f_n \ge 0$, then $\sum_{1}^{\infty} \int_X f_n d\mu = \int_X (\sum_{1}^{\infty} f_n) d\mu$.

Theorem 7.18. If $f : \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, then the Riemann integral of f is the same as the Lebesgue integral. Proof. IOU

Question 7.19. Let $f: [0, \infty) \to \mathbb{R}$ be measurable, and define the Laplace transform of f by $F(s) = \int_0^\infty e^{-st} f(t) dt$. Is F continuous? Is F differentiable?

Question 7.20. Let φ be a bump function, and (q_n) be an enumeration of the rationals. Define $f(x) = \sum_{n=1}^{\infty} \varphi(2^n(x-q_n))$. Is f finite almost everywhere?

7.3. Push forward measures.

Definition 7.21. Say $f: X \to \mathbb{R}^d$ is integrable, then define $\int_X f \, d\mu = (\int_X f_1 \, d\mu, \dots, \int_X f_d \, d\mu$, where $f = (f_1, \dots, f_d)$.

Theorem 7.22. Let (X, Σ, μ) be a measure space, $f: X \to Y$ be arbitrary. Define $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$, and define $\nu(A) = \mu(f^{-1}(A))$. Then ν is a measure on (Y, τ) and $\int_Y g \, d\nu = \int_X g \circ f \, d\mu$.

Remark 7.23. The measure ν is called the *push forward* of μ and denoted by $f^*(\mu)$, or $\mu_{f^{-1}}$. This is used often to define Laws of random variables. (We will use it to prove the change of variable formula.)

Corollary 7.24. If $\alpha \in \mathbb{R}^d$, then $\int_{\mathbb{R}^d} f(x+\alpha) d\lambda(x) = \int_{\mathbb{R}^d} f(x) d\lambda(x)$.

8. Convergence

8.1. Modes of convergence.

Definition 8.1. We say $(f_n) \to f$ almost everywhere if for almost every $x \in X$, we have $(f_n(x)) \to f(x)$.

Definition 8.2. We say $(f_n) \to f$ in measure (notation $(f_n) \xrightarrow{\mu} f$) if for all $\varepsilon > 0$, we have $(\mu\{|f_n - f| > \varepsilon\}) \to 0$.

Definition 8.3. Let $p \in [1, \infty)$. We say $(f_n) \to f$ in L^p if $(\int_X |f_n - f|^p d\mu) \to 0$.

Question 8.4. Why p > 1? How about $p = \infty$?

(1) (f_n) → f almost everywhere implies (f_n) → f in measure if µ(X) < ∞.
(2) (f_n) → f in measure implies (f_n) → f almost everywhere along a subsequence.
(3) (f_n) → f in L^p implies (f_n) → f in measure (for p < ∞), and hence (f_n) → f along a subsequence.
(4) Convergence almost everywhere or in measure don't imply convergence in L^p.

Theorem 8.5. If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, then $(f_n) \to f$ in measure.

Lemma 8.6 (Egorov). If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, for every $\varepsilon > 0$ there exists A_{ε} such that $\mu(A_{\varepsilon}^c) < \varepsilon$ and $(f_n) \to f$ uniformly on A_{ε} .

Question 8.7. Does this imply $(f_n) \rightarrow f$ uniformly almost everywhere?

Proof of Theorem 8.5

Proposition 8.8. If $(f_n) \to f$ in measure then (f_n) need not converge to f almost everywhere.

Proposition 8.9. If $(f_n) \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere. 8.2. L^p spaces.

Definition 8.10. A *Banach space* is a normed vector space that is complete under the metric induced by the norm. *Example* 8.11. \mathbb{C} , \mathbb{R}^d , C(X), etc. are all Banach spaces.

Definition 8.12. For $p \in (0, \infty)$, define $||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p}$.

Definition 8.13. For $p = \infty$, define $||f||_{\infty} = \operatorname{ess\,sup}|f| = \inf\{C \ge 0 \mid |f| \le C \text{ almost surely}\}$

Definition 8.14. Let (X, Σ, μ) be a measure space, and assume Σ is μ -complete. Define $\mathcal{L}^p(X) = \{f : X \to \mathbb{R} \mid ||f||_p < \infty\}$. **Question 8.15.** Is $\mathcal{L}^p(X)$ a Banach space?

Definition 8.16. Define an equivalence relation on \mathcal{L}^p by $f \sim g$ if f = g almost everywhere.

Definition 8.17. Define $L^p(X) = \mathcal{L}^p(X) / \sim$.

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation \sim . In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation \sim .

Theorem 8.19. For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.

Theorem 8.20 (Hölder's inequality). Say $p, q \in [1, \infty]$ with 1/p + 1/q = 1. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $|\int_X fg \, d\mu| \leq ||f||_p ||g||_q$.

Remark 8.21. The relation between p and q can be motivated by dimension counting, or scaling.

Brute force proof of Theorem 8.20 Proof of Theorem 8.20 using Young's inequality.

Theorem 8.22 (Young's inequality). If $x, y \ge 0$, 1/p + 1/q = 1 then $xy \le x^p/p + y^q/q$.

Lemma 8.23 (Duality). If
$$p \in [1, \infty)$$
, $1/p + 1/q = 1$, then $||f||_p = \sup_{g \in L^q - 0} \frac{1}{||g||_q} \int_X fg \, d\mu = \sup_{||g||_q = 1} \int_X fg \, d\mu = 1$

Remark 8.24. For $p = \infty$ this is still true if X is σ -finite.

Theorem 8.25 (Minkowski's inequality). If $f, g \in L^p$, then $f + g \in L^p$ and $||f + g||_p \leq ||f||_p + ||g||_p$.

Theorem 8.26 (Jensen's inequality). If $\mu(X) = 1$, $f \in L^1(X)$, a < f < b almost everywhere, and $\varphi : (a, b) \to \mathbb{R}$ is convex, then

$$\varphi\Big(\int_X f\,d\mu\Big) \leqslant \int_X \varphi \circ f\,d\mu$$

Proof of Theorem 8.19: Only remains to show L^p is complete.

Lemma 8.27. Suppose $p < \infty$, $f_n \in L^p$ and $\sum ||f_n||_p < \infty$. Let $f = \sum f_n$. Then $f \in L^p$, and $\sum f_n \to f$ in L^p and $\sum f_n \to f$ almost everywhere.

Proof of Theorem 8.19:

Proposition 8.28. If $p \in [1, \infty)$, $(f_n) \to f$ in L^p , then $(f_n) \to f$ in measure.

Lemma 8.29 (Chebychev's inequality). For any $\lambda > 0$, we have $\mu(\{|f| > \lambda\}) \leq \frac{1}{\lambda} ||f||_1$

Proof of Proposition 8.28

8.3. Uniform integrability.

Question 8.30. When does convergence in measure imply L^1 convergence?

Definition 8.31. We say $\{f_{\alpha}\alpha \in \mathcal{A}\}$ is uniformly integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$ we have $\int_{E} |f| d\mu < \varepsilon$.

Proposition 8.32. If $|f_{\alpha}| \leq F$ for all $\alpha \in A$, and $F \in L^1$, then $\{f_{\alpha} \mid \alpha \in A\}$ is uniformly integrable.

Theorem 8.33 (Vitali). Let $(f_n) \in L^1(X)$. The sequence (f_n) is convergent in L^1 if and only if

- (1) (f_n) converges in measure,
- (2) (f_n) is uniformly integrable,
- (3) (tightness) there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\int_{F^c} |f_n| d\mu < \varepsilon$ for all n.

Proof:

Theorem 8.34. If
$$\lim_{\lambda \to \infty} \sup_{n} \int_{\{|f_n| > \lambda\}} |f_n| d\mu = 0$$
, then (f_n) is uniformly integrable.

Theorem 8.35. If there exists an increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty$, and $\sup_n \int_X \varphi(|f_n|) d\mu < \infty$, then (f_n) is uniformly integrable.

Remark 8.36. The hypothesis in both the above theorems are equivalent.

Remark 8.37. If additionally $\sup_n \int_X |f_n| d\mu < \infty$, then the converse of both the above theorems are true. *Proof:*

Corollary 8.38. If $(f_n) \to f$ in measure, $\mu(X) < \infty$, and $\sup_n \|f\|_p < \infty$ for any p > 1, then $(f_n) \to f$ in L^q for every $q \in [1, p)$.

9. Signed Measures

9.1. Hanh and Jordan Decomposition Theorems.

Definition 9.1. We say $\mu \colon \Sigma \to [-\infty, \infty]$ is a signed measure if:

- (1) The range of μ doesn't contain both $+\infty$ and $-\infty$.
- (2) $\mu(\emptyset) = 0$
- (3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Example 9.2. Let $f \in L^1(X,\mu)$, and define ν by $\nu(A) = \int_A f d\mu$. Then ν is a signed measure, and we write $d\nu = f d\mu$.

Example 9.3. If μ , ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is finite.

Theorem 9.4 (Jordan Decomposition). Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leq 0$ for all measurable sets $B \subseteq A$.

Proposition 9.6. If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and $\mu(B) \leq \mu(A)$.

Theorem 9.7 (Hanh decomposition). If μ is a signed measure on X, then $X = P \cup N$ where P is positive and N is negative.

Remark 9.8. The decomposition is unique up to null sets.

Definition 9.9. We say two positive measures μ, ν are *mutually singular* if there exists $C \subseteq X$ such that for every $A \in \Sigma$ we have $\mu(A \cap C) = \nu(A \cap C^c) = 0$.

Proof of Theorem 9.4

Definition 9.10. Let μ be a signed measure with Jordan decomposition $\mu = \mu^+ - \mu^-$. Define the variation of μ to be the (positive) measure $|\mu| \stackrel{\text{def}}{=} \mu^+ + \mu^-$.

Definition 9.11. Define the *total variation* of μ by $\|\mu\| = |\mu|(X)$.

Proposition 9.12. Let \mathcal{M} be the set of all finite signed measures on X. Then \mathcal{M} is a Banach space under the total variation norm.

9.2. Absolute Continuity.

Definition 9.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \ge 0$ and define $\nu(A) = \int_A g \, d\mu$. (Notation: Say $d\nu = g \, d\mu$.)

Theorem 9.15 (Radon-Nikodym). If μ, ν are two positive σ -finite measures such $\nu \ll \mu$, then there exists a unique measurable function g such that $0 \leq g < \infty$ almost everywhere and $d\nu = g d\mu$.

Theorem 9.16. Let μ, ν be positive measures such that ν is σ -finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu, \nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.

Corollary 9.17. Let μ be a positive measure, and ν be a finite signed measure. There exists a unique pair of signed measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$ and $\nu = \nu_{ac} + \nu_s$.

Corollary 9.18. Let μ, ν be σ -finite positive measures. There exists a unique positive measure ν_s and nonnegative measurable function g such that $\mu \perp \nu_s$ and $d\nu = d\nu_s + g d\mu$.

9.3. Dual of L^p .

Proposition 9.19. Let U, V be Banach spaces, and $T: U \to V$ be linear. Then T is continuous if and only if there exists $c < \infty$ such that $||Tu||_V \leq c||u||_U$ for all $u \in U$, $v \in V$.

Definition 9.20. We say $T: U \to V$ is a bounded linear transformation if T is linear and there exists $c < \infty$ such that $||Tu||_V \leq c||u||_U$ for all $u \in U, v \in V$.

Definition 9.21. The dual of U is defined by $U^* = \{u^* \mid u^* \colon U \to \mathbb{R} \text{ is bounded and linear.}\}$ Define a norm on U^* by

$$\|u^*\|_{U^*} \stackrel{\text{def}}{=} \sup_{u \in U-0} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U=1} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U=1} \frac{1}{\|u\|_U} |u^*(u)|.$$

Proposition 9.22. The dual of a Banach space is a Banach space.

Proposition 9.23. Let 1/p + 1/q = 1, $g \in L^q(X)$. Define $T_g \colon L^p \to \mathbb{R}$ by $T_g f = \int_X fg \, d\mu$. Then $T_g \in (L^p)^*$.

Proposition 9.24. The map $g \mapsto T_g$ is a bounded linear map from $L^q \to (L^p)^*$.

Theorem 9.25. Let (X, Σ, μ) be a σ -finite measure space, $p \in [1, \infty)$, 1/p + 1/q = 1. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ -finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_q$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

9.4. Riesz Representation Theorem.

Theorem 9.28 (Riesz Representation Theorem). Let X be a compact metric space, and \mathcal{M} be the set of all finite signed measures on X. Define $\Lambda: \mathcal{M} \to C(X)^*$ by $\Lambda_{\mu}(f) = \int_X f \, d\mu$ for all $\mu \in \mathcal{M}$ and $f \in C(X)$. Then Λ is a bijective linear isometry.

Remark 9.29. In particular, for every $I \in C(X)^*$, there exists a unique finite regular Borel measure μ such that $I(f) = \int_X f d\mu$ for every $f \in C(X)$.

10. Product measures

10.1. Fubini and Tonelli theorems. Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma, B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 10.1. Let μ, ν be two σ -finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.

Theorem 10.2 (Tonelli). Let $f: X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$ -measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,

(10.1)
$$\int_{X \times Y} f(x, y) \, d\pi(x, y) = \int_{x \in X} \left(\int_{y \in Y} f(x, y) \, d\nu(y) \right) d\mu(x) = \int_{y \in Y} \left(\int_{x \in X} f(x, y) \, d\mu(x) \right) d\nu(y) \, .$$

Theorem 10.3 (Fubini). If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover, (10.1) holds.

Lemma 10.4. For every $E \subseteq X \times Y$, $x \in X$, $y \in Y$ define the horizontal and vertical slices of E by $H_y(E) = \{x \in X \mid (x, y) \in E\}$ and $V_x(E) = \{y \in Y \mid (x, y) \in E\}$.

- (1) For every $x \in X$, $y \in Y$ we have $H_y(E) \in \Sigma$ and $V_x(E) \in \tau$.
- (2) The functions $x \mapsto \nu(V_x(E))$ and $y \mapsto \mu(H_y(E))$ are measurable.

Proof of Theorem 10.1 Proof of Theorem 10.2 Proof of Theorem 10.3

Theorem 10.5 (Layer Cake). If $f: X \to [0, \infty]$ is measurable then $\int_X f \, d\mu = \int_0^\infty \mu(f > t) \, dt$.

Proposition 10.6. If $(a_{m,n})$ are such that $\sum_{m,n=0}^{\infty} |a_{m,n}| < \infty$, then $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n}$.

Theorem 10.7 (Minkowski's inequality). If $f: X \times Y \to \mathbb{R}$ is measurable, then

$$\left(\int_X \left|\int_Y f(x,y) \, d\nu(y)\right|^p d\mu(x)\right)^{1/p} \leqslant \int_Y \left(\int_X |f(x,y)|^p \, d\mu(x)\right)^{1/p} d\nu(y)$$

10.2. Convolutions.

Definition 10.8. If
$$f, g \in L^1(\mathbb{R}^d)$$
 define the convolution by $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \, dy = \int_{\mathbb{R}^d} f(y)g(x-y) \, dy$.

Remark 10.9. If $f, g \in L^1(\mathbb{R}^d)$, then $f * g < \infty$ almost everywhere.

Theorem 10.10 (Young). If $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^{p}(\mathbb{R}^{d})$, $g \in L^{q}(\mathbb{R}^{d})$ then $f * g \in L^{r}(\mathbb{R}^{d})$, and $||f * g||_{L^{r}} \leq ||f||_{L^{p}} ||g||_{L^{q}}$.

Remark 10.11. One can show $||f * g||_r \leq C_{p,q} ||f||_p ||g||_q$ for some constant $C_{p,q} < 1$. The optimal constant can be found by choosing f, g to be Gaussian's.

Definition 10.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \ge 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\lim_{n \to \infty} \int_{\{|y| > \varepsilon\}} \varphi_n(y) \, dy = 0$.

Example 10.13. Let $\varphi \ge 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_{\varepsilon} = \frac{1}{\varepsilon^d} \varphi(\frac{x}{\varepsilon})$. Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$, for $x \in \mathbb{R}^d$.

Proposition 10.15. If $p \in [1, \infty)$, $f \in L^p$, and (φ_n) is an approximate identity, then $\varphi_n * f \to f$ in L^p . Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous. 10.3. Fourier Series. Let X = [0, 1] with the Lebesgue measure. For $n \in \mathbb{Z}$ define $e_n(x) = e^{2\pi i n x}$, and given $f, g \in L^2(X, \mathbb{C})$ define $\langle f, g \rangle = \int_X f \bar{g} \, d\lambda$. This defines an *inner product* on $L^2(X)$, and $\|f\|_{L^2}^2 = \langle f, f \rangle$.

Definition 10.17. If $f \in L^2$, $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f by $\hat{f}(n) = \langle f, e_n \rangle$.

Definition 10.18. For $N \in \mathbb{N}$, let $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n$, be the *N*-th partial sum of the *Fourier Series* of *f*. **Question 10.19.** Does $S_N f \to f$? In what sense?

Lemma 10.20. $\langle e_n, e_m \rangle = \delta_{n,m}$.

Corollary 10.21. Let $p \in \text{span}\{e_{-N}, \dots, e_N\}$. Then $\langle f - S_N f, p \rangle = 0$. Consequently, $||f - S_N f||_2 \leq ||f - p||_2$.

Proposition 10.22. $S_N f = D_N * f$, where $D_N = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(\pi x)}$. The functions D_N are called the Dirichlet Kernels.

Proposition 10.23. Define the Cesàro sum by $\sigma_N f = \frac{1}{N} \sum_{0}^{N-1} S_n f$. Then $\sigma_N f = F_N * f$, where $F_N = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2$.

 \square

Remark 10.24. The functions F_N are called the Fejér Kernels.

Proposition 10.25. The Fejér kernels are an approximate identity, but the Dirichlet kernels are not.

Corollary 10.26. If $p \in [1, \infty)$ and $f \in L^p$, then $\sigma_N f \to f$ in L^p .

Corollary 10.27. If $f \in L^2$ then $S_N f \to f$ in L^2 .

Theorem 10.28. If $p \in (1, \infty)$, $f \in L^p$ then $S_N f \to f$ in L^p .

Proof. The proof requires boundedness of the Hilbert transform and is beyond the scope of this course.

Theorem 10.29. If $f \in L^{\infty}$ and is Hölder continuous at x with any exponent $\alpha > 0$, then $S_n f(x) \to x$.

Proof. On homework.

Remark 10.30. If f is simply continuous at x, then certainly $\sigma_n f(x) \to f(x)$, but $S_n f(x)$ need not converge to f(x). In fact, for almost every continuous periodic function, $S_N f$ diverges on a dense G_{δ} .

The next few results establish a connection between the regularity (differentiability) of a function and decay of its Fourier coefficients.

Theorem 10.31 (Riemann Lebesgue). Let μ be a finite measure and set $\hat{\mu}(n) = \int_0^1 \bar{e_n} d\mu$. If $\mu \ll \lambda$, then $(\hat{\mu}(n)) \to 0$ as $n \to \infty$. **Theorem 10.32** (Parseval's equality). If $f \in L^2([0,1])$ then $\|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$.

Question 10.33. What are the Fourier coefficients of f'?

Definition 10.34. We say g is a weak derivative of f if $\langle f, \varphi' \rangle = -\langle g, \varphi \rangle$ for all $\varphi \in C^{\infty}_{per}([0,1])$.

Proposition 10.35. If $f \in L^1$ has a weak derivative $f' \in L^1$, then $(f')^{\wedge}(n) = 2\pi i n \hat{f}(n)$.

Corollary 10.36. If $f \in L^2$ has a weak derivative $f' \in L^2$, then $\sum [(1+|n|)|\hat{f}(n)|]^2 < \infty$.

Definition 10.37. For $s \ge 0$, let $H_{per}^s \stackrel{\text{def}}{=} \{f \in L^2 \mid ||f||_{H^s} < \infty\}$, where $||f||_{H^s}^2 = \sum (1+|n|)^{2s} |\hat{f}(n)|^2$.

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2 .

Theorem 10.39 (1D Sobolev Embedding). If $s > \frac{1}{2}$ and $H_{per}^s \subseteq C_{per}([0,1])$ and the inclusion map is continuous. Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when s = 1/2. Remark 10.41. In d dimensions the above is still true if you assume s > d/2.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H_{per}^s \subseteq C^{n,\alpha}$.

Theorem 10.43 (1D Sobolev embedding). If $s > \frac{1}{2} - \frac{1}{2n}$, then $H_{per}^s \subseteq L^{2n}$ and the inclusion map is continuous.

Remark 10.44. The above is true for $s = \frac{1}{2} - \frac{1}{p}$, for some $p \in [1, \infty)$ but our proof won't work.

11. Differentiation

11.1. Lebesgue Differentiation.

Theorem 11.1 (Fundamental theorem of Calculus 1). If f is continuous and $F(x) = \int_0^x f(t) dt$, then F is differentiable and F' = f. **Theorem 11.2** (Fundamental theorem of Calculus 2). If f is Riemann integrable, and F' = f, then $\int_a^b f = F(b) - F(a)$. Our goal is to generalize these to Lebesgue integrable functions.

Theorem 11.3 (Lebesgue Differentiation). If $f \in L^1(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^d$, $\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f \, d\lambda = f(x)$.

Lemma 11.4 (Vitali Covering Lemma). Let $W \subseteq \bigcup_{i=1}^{N} B(x_i, r_i)$. There exists $S \subseteq \{1, \ldots, N\}$ such that:

(1) $\{B(x_i, r_i) \mid i \in S\}$ are pairwise disjoint. (2) $W \subseteq \bigcup_{i \in S} B(x_i, 3r_i)$ and hence $|W| \leq 3^d \sum_{i \in S} B(x_i, r_i)$.

Definition 11.5 (Maximal function). Let μ be a finite (signed) Borel measure on \mathbb{R}^d . Define the maximal function of μ by

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}$$

Proposition 11.6. $M\mu \in L^{1,\infty}$, and $|M\mu > \alpha| \leq \frac{3^d}{\alpha} ||\mu||$.

Corollary 11.7. If $f \in L^1(\mathbb{R}^d)$, then $|\{Mf > \alpha\}| \leq \frac{3^d}{\alpha} ||f||_{L^1}$.

Proposition 11.8. If $f \in L^1(\mathbb{R}^d)$, then $\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{|y-x| < r} |f(y) - f(x)| dy = 0$ almost everywhere.

Remark 11.9. This immediately implies Theorem 11.3.

Corollary 11.10. If $\mu \ll \lambda$ is a finite signed measure, then the Radon-Nikodym derivative is given by $\frac{d\mu}{d\lambda} = \lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|}$. Remark 11.11. Will use this to prove the change of variables formula.

11.2. Fundamental theorem of calculus.

Question 11.12. Does $f: [0,1] \to \mathbb{R}$ differentiable almost everywhere imply $f' \in L^1$?

Question 11.13. Does $f: [0,1] \to \mathbb{R}$ differentiable almost everywhere, and $f' \in L^1$ imply $f(x) = \int_0^x f'$?

Definition 11.14. We say $f: \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{N} |x_i - y_i| < \delta \implies \sum_{i=1}^{N} |f(x_i) - f(y_i)| < \varepsilon$.

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.

Theorem 11.16. Let $f: [a,b] \to \mathbb{R}$ be measurable. Then f is absolutely continuous if and only if f is differentiable almost everywhere, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f'$ everywhere.

Proof of the reverse implication of Theorem 11.16

Lemma 11.17. If f is absolutely continuous, monotone and injective, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ everywhere.

Lemma 11.18. If f is absolutely continuous and monotone, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f' almost$ everywhere.

 \Box

Lemma 11.19. If f is absolutely continuous then there exist g, h increasing such that f = g - h.

Proof of the forward implication of Theorem 11.16. Follows immediately from the previous lemmas.

11.3. Change of variables.

Theorem 11.20. Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi \colon U \to V$ be C^1 and bijective. If $f \in L^1(V)$, then $\int_V f \, d\lambda = \int_U f \circ \varphi |\det \nabla \varphi| \, d\lambda$.

The main idea behind the proof is as follows: Let $\mu(A) = \lambda(\varphi(A))$. Lemma 11.21. μ is a Borel measure and $\int_U f \circ \varphi \, d\mu = \int_V f \, d\lambda$. Lemma 11.22. $\mu \ll \lambda$

Lemma 11.23. $D\mu = |\det \nabla \varphi|$, where $D\mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|}$.

Proof of Theorem 11.20. Follows immediately from the above Lemmas.

Proof of Lemma 11.21 Proof of Lemma 11.22 Proof of Lemma 11.23

12. Fourier Transform

12.1. Definition and Basic Properties.

(1) Recall if $f \in L^2_{per}([0,1])$, we set $e_n(x) = e^{2\pi i n x}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x} dx$ and got $f = \sum a_n e_n$ in L^2 . (2) Suppose now $f \in L^2_{per}([-L/2, L/2])$. Can we rescale and send $L \to \infty$?

Definition 12.1. If $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, define the Fourier transform of f (denoted by \hat{f}) by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$

Remark 12.2. More generally, if μ is a finite (signed) Borel measure, then can define $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle} d\mu(x).$

Analogous to Fourier series, we will show that \hat{f} is defined even for $f \in L^2$, and prove $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} d\xi$.

Lemma 12.3 (Linearity). If $f, g \in L^1$, $\alpha \in \mathbb{R}$ then $(f + \alpha g)^{\wedge} = \hat{f} + \alpha \hat{g}$. Lemma 12.4 (Translations). Let $\tau_y f(x) = f(x - y)$. Then $(\tau_y f)^{\wedge}(\xi) = e^{-2\pi i \langle y, \xi \rangle} \hat{f}(\xi)$. Lemma 12.5 (Dilations). Let $\delta_{\lambda} f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. Then $(\delta_{\lambda} f)^{\wedge}(\xi) = \hat{f}(\lambda \xi)$. Lemma 12.6. If $f, g \in L^1$, then $(f * g)^{\wedge} = \hat{f}\hat{g}$. Lemma 12.7. If $(1 + |x|)f(x) \in L^1(\mathbb{R}^d)$ then $\partial_j \hat{f}(\xi) = (-2\pi i x_j f(x))^{\wedge}(\xi)$. Lemma 12.8. If $f \in C_0^1$, $\partial_j f \in L^1$, then $(\partial_j f)^{\wedge}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$. Theorem 12.9 (Riemann-Lebesgue Lemma). If $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_{L^{\infty}} \leq \|f\|_{L^1}$.

12.2. Fourier Inversion.

Theorem 12.10 (Inversion). If
$$f, \hat{f} \in L^1$$
, then $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} d\xi$.

Direct proof attempt:

Lemma 12.11. If $G(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, then $\hat{G}(\xi) = e^{-|2\pi\xi|^2/2}$, and hence $\hat{G} = G$. **Lemma 12.12.** If $f, g \in L^1$ then $\int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}^d} \hat{f}g$.

Lemma 12.13. If $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} d\xi$.

Proof of Theorem 12.10.

Remark 12.14. If
$$f, \hat{f} \in L^1$$
, then $||f - \varphi_{\varepsilon} * f||_{L^{\infty}} \leq ||\hat{f} - (\varphi_{\varepsilon} * f)^{\wedge}||_{L^1} \to 0$
Remark 12.15. If $f, \hat{f} \in L^1$ then $\hat{f}(x) = f(-x)$.

12.3. L^2 -theory.

Theorem 12.16 (Plancherel). The Fourier transform extends to a bijective linear isometry on $L^2(\mathbb{R}^d;\mathbb{C})$.

Definition 12.17. Define the Schwartz space, S, to be the set of all smooth functions such that $\sup_x (1+|x|^n)|D^{\alpha}f(x)| < \infty$ for all $n \in \mathbb{N}$ and multi-indexes α .

Remark 12.18. Note $C_c^{\infty}(\mathbb{R}^d) \subseteq S$, and so S is a dense subset of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

Lemma 12.19. If $f, g \in S$, then $\int_{\mathbb{R}^d} f \bar{g} dx = \int_{\mathbb{R}^d} \hat{f} \hat{g} d\xi$.

Proof of Theorem 12.16

Definition 12.20. Let $s \ge 0$ and define the Sobolev space of index s by

$$H^{s} = \{ f \in L^{2}(\mathbb{R}^{d}) \mid \|f\|_{H^{s}} < \infty \}, \quad \text{where} \quad \|f\|_{H^{s}} = \left(\int_{\mathbb{R}^{d}} (1 + |\xi|^{2})^{s} |\hat{f}(\xi)|^{2} d\xi \right)^{1/2}.$$

Remark 12.21. A function $f \in H^1$ if and only if f and all first order weak derivatives are in L^2 .

Remark 12.22. For s < 0, one needs to define H^s as the completion of S under the H^s norm.

Proposition 12.23. Let $s \in (0,1)$. Then $f \in H^s$ if and only if $\int_0^\infty \left(\frac{\|f - \tau_{hv}f\|_{L^2}}{|h|^s}\right)^2 \frac{dh}{h^d} < \infty$ for all $v \in \mathbb{R}^d$. Remark 12.24. For s = 1, we instead need $\sup_{h>0} \frac{1}{h} \|f - \tau_{hv}f\|_{L^2} < \infty$. Remark 12.25. If $s \in (0,1]$, then there exists C = C(s) such that $\|f - \tau_h f\|_{L^2} \leq C|h|^s \|f\|_{L^2}$ for all $f \in H^s$, $h \in \mathbb{R}^d$. Theorem 12.26 (Sobolev embedding). If s > d/2 then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous. Corollary 12.27. If s > n + d/2, then $H^s(\mathbb{R}^d) \subseteq C_b^n(\mathbb{R}^d)$ and the inclusion map is continuous. Proposition 12.28 (Elliptic regularity). Say $f \in S(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ is such that $\lim_{|x|\to\infty} |x|^d |\nabla u(x)| = 0$ and $-\Delta u = f$, then $u \in S$.

Appendix A. The *d*-dimensional Hausdorff measure in \mathbb{R}^d

Let (X, d) be any metric space, $\delta > 0$, $\alpha \ge 0$ and $H^*_{\alpha, \delta}$ be the outer measure defined by

$$H^*_{\alpha,\delta}(A) = \inf\left\{\sum_{1}^{\infty} \rho_{\alpha}(E_i) \mid \operatorname{diam}(E_i) < \delta, \text{ and } A \subset \bigcup_{1}^{\infty} E_j\right\}, \text{ where } \rho_{\alpha}(A) = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})} \left(\frac{\operatorname{diam}(A)}{2}\right)^{\alpha}.$$

Remark A.1. The function ρ_{α} above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_{\alpha} = \lim_{\delta \to 0} H^*_{\alpha}$.

Proposition A.3 (From homework 2). The outer measure H^*_{α} restricts to a measure on the Borel σ -algebra.

Theorem A.4. If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_{\alpha} = \lambda$ (the Lebesgue measure).

Lemma A.5 (Infinite version of Vitali's Covering Lemma). Let $W \subseteq \bigcup_{\alpha \in A} B(x_{\alpha}, r_{\alpha})$, with $\sup r_{\alpha} < \infty$. There exists a countable set $\mathcal{I} \subseteq A$ such that:

- (1) $\{B(x_i, r_i) \mid i \in \mathcal{I}\}$ are pairwise disjoint.
- (2) $W \subseteq \bigcup_{i \in \mathcal{I}} B(x_i, 5r_i)$ and hence $|W| \leq 5^d \sum_{i \in S} B(x_i, r_i)$.

Lemma A.6. Let $U \subseteq \mathbb{R}^d$ be open and $\delta > 0$. There exists countably many $x_i \in U$, $r_i \in (0, \delta)$ such that $\overline{B(x_i, r_i)} \subseteq U$, are pairwise disjoint, and $|U - \bigcup \overline{B(x_i, r_i)}| = 0$.

Lemma A.7. $H_d \leq \lambda$.

Theorem A.8 (Isodiametric inequality). $|A| \leq |B(0, 1/2)| \operatorname{diam}(A)^d = |B(0, \operatorname{diam}(A)/2)|$.

Remark A.9. Note A need not be contained in a ball of radius diam(A)/2.

Proof of Theorem A.4.

Proposition A.10 (Steiner Symmetrization). Let $P \subseteq \mathbb{R}^d$ be a hyperplane with unit normal \hat{n} . Let $A \in \mathcal{L}(\mathbb{R}^d)$. There exists $S_P(A) \in \mathcal{L}(\mathbb{R}^d)$ such that:

- (1) $S_P(A)$ is symmetric about P (i.e. for any $x \in P$, $t \in \mathbb{R}$, we have $x + t\hat{n} \in S_P(A) \iff x t\hat{n} \in S_P(A)$).
- (2) diam $(S_P(A)) \leq$ diam(A).

(3) $|S_P(A)| = |A|.$

Proof of Theorem A.8