LECTURE NOTES ON MEASURE THEORY
FALL 2020

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Contents

1. **Preface.**

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LaTeXsource of these slides is also available on git.

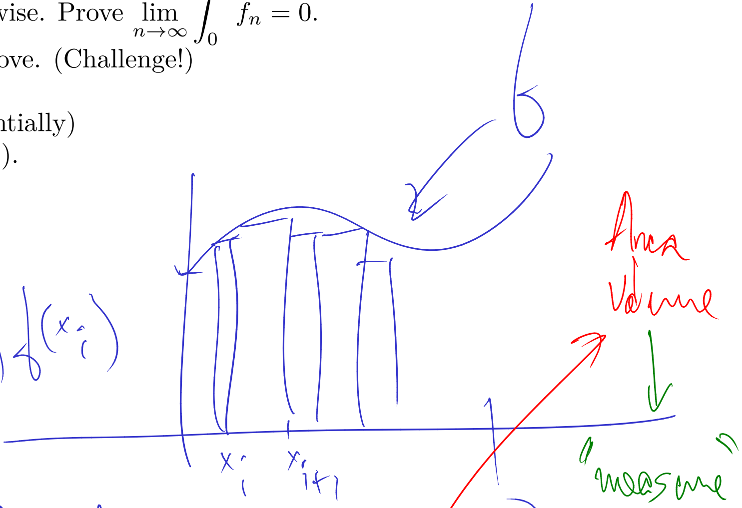
1. Syllabus Overview

- Class website and full syllabus: `http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/720-measure`
- TA: Lantian Xu <lxu2@andrew.cmu.edu>
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterm: Fri Oct 9th (90 mins, self proctored, can be taken any time)
- **Zoom lectures:**
  - ▷ Please enable video. (It helps me pace lectures).
  - ▷ Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - ▷ If I get disconnected, check your email for instructions.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Bottom 20% homework is dropped from your grade (personal emergencies, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - ▷ Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - ▷ Self proctored: Zoom call (invite me). Record yourself, and your screen to the cloud.
  - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.

- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
    - Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 40% homework, 20% midterm, 40% final.

2. Sigma Algebras and Measures

- **Motivation:** Suppose $f_n \colon [0,1] \to [0,1]$, and $(f_n) \to 0$ pointwise. Prove $\lim_{n\to\infty} \int_0^1 f_n = 0$.
  - ▷ Simple to state using Riemann integrals. Not so easy to prove. (Challenge!)
  - ▷ Will prove this using Lebesgue integration.
    - Riemann integration: partition the domain (count sequentially)
    - Lebesgue integration: partition the range (stack and sort).
- **Goal:**
  - ▷ Develop Lebesgue integration.
  - ▷ Need a notion of "measure" (generalization of volume)
  - ▷ Need "σ-algebras".



Riemann: $\int_0^1 f \approx \sum (x_{i+1} - x_i) f(x_i)$

Lebesgue: Say $f = \begin{cases} a_1 & \text{on } A_1 \\ a_2 & \text{on } A_2 \end{cases}$ $A_1, A_2, \ldots$ disj.

$\rightsquigarrow \int_0^1 f \approx \sum a_i \text{ "length"}(A_i)$

$n < 5$

- Why σ-algebras?

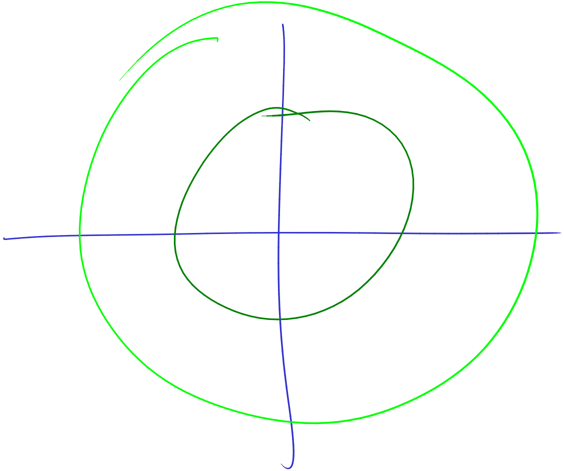
Theorem 2.1 ((Banach Tarski)). *There exists $n \in \mathbb{N}$, sets $A_1, \ldots, A_n \subseteq B(0,1) \subseteq \mathbb{R}^3$ such that:*

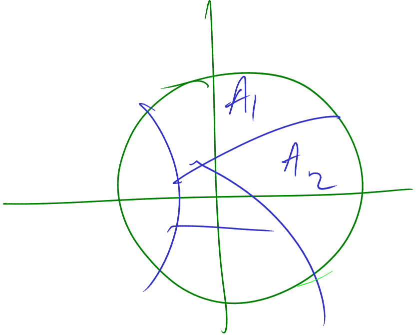
(1) $A_1, \ldots, A_n$ partition $B(0,1)$.

(2) There exist isometries R_i such that $R_1(A_1), \ldots, R_n(A_n)$ partition $B(0,2)$.

$\mathbb{R}^3$

- How do you explain this?





Definition 2.2 (σ-algebra)**.** Let X be a set. We say $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra on X if:

1. Nonempty: $\emptyset \in \Sigma$
2. Closed under compliments: $A \in \Sigma \implies A^c \in \Sigma$.
3. Closed under countable unions: $A_i \in \Sigma \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

$A_1, A_2, \dots \in \Sigma \Rightarrow \bigcup_i^{\infty} A_i \in \Sigma$

Remark 2.3. Any σ-algebra is also closed under countable intersections.

Demogan's law.

Question 2.4. *Is $\mathcal{P}(X)$ is a σ-algebra?*

✓ **Question 2.5.** *Is $\Sigma \overset{\text{def}}{=} \{\emptyset, X\}$ is a σ-algebra?*

Question 2.6. *Is $\Sigma = \{A \mid |A| < \infty$ or $|A^c| < \infty\}$ a σ-algebra?* NO

X is inf.

Question 2.7. *Is $\Sigma = \{A \mid$ either A or A^c is finite or countable$\}$ a σ-algebra?*

Yes.

(finite is countable)

Proposition 2.8. *If* $\forall \alpha \in \mathcal{A}$, Σ_α *is a* σ*-algebra, then so is* $\bigcap_{\alpha \in \mathcal{A}} \Sigma_\alpha$. Yes.

Definition 2.9. If $\mathcal{E} \subseteq \mathcal{P}(X)$, define $\boxed{\sigma(\mathcal{E})}$ to be the intersection of all σ-algebras containing $\mathcal{E}$.

Remark 2.10. $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.

Definition 2.11. Suppose X is a topological space. The *Borel* σ*-algebra on* X is defined to be the σ-algebra generated by all open subsets of X. Notation: $\mathcal{B}(X)$.

Question 2.12. *Can you get* $\mathcal{B}(X)$ *by taking all countable unions / intersections of open and closed sets?*

Question 2.13. *Is* $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$*?*

$X = \mathbb{R}$. $\mathcal{E} = \{\text{open sets}\}$. $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$

Guess: $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R}) = \{\text{open sets}\} \cup \{\text{closed sets}\} \cup G_\delta \cup F_\sigma \cup G_{\delta\sigma} \cup F_{\sigma\delta} \cup \dots$

(Not enough!)

Definition 2.14. Let Σ be a σ-algebra on X. We say μ is a (~~positive~~) measure on (X, Σ) if:

1. $\mu\colon \Sigma \to [0, \infty]$
2. $\mu(\emptyset) = 0$
3. (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 2.15. *Is the second assumption necessary?*

Question 2.16. *Let $\mu(A) =$ cardinality of A. Is μ a measure?*

Question 2.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?*

Theorem 2.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \mathrm{vol}(I)$ for all cuboids I.*

~~A~~ $A \in \Sigma \Rightarrow A \subseteq X, \mu(A) \in [0, \infty]$

$E \in \Sigma$.

$$\cancel{\mu(E)} = \mu(E \cup \emptyset) = \cancel{\mu(E)} + \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0$$

Definition 3.14. Let Σ be a σ-algebra on X. We say μ is a (positive) measure on (X, Σ) if:

1. $\mu\colon \Sigma \to [0, \infty]$
2. $\mu(\emptyset) = 0$
3. (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 3.15. *Is the second assumption necessary?*

YES

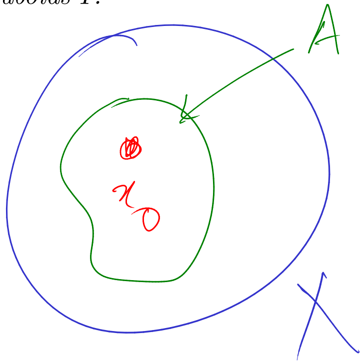
Question 3.16. *Let $\mu(A) =$ cardinality of A. Is μ a measure?*

$$\mu(A) = \begin{cases} \#\text{ elms in } A & (\text{if } A \text{ is fin}) \\ +\infty \end{cases}$$

Question 3.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?*

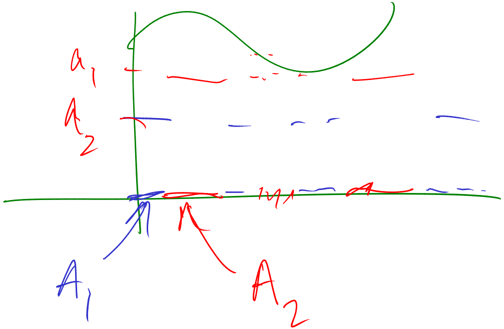
⟵ δ measure (YES).

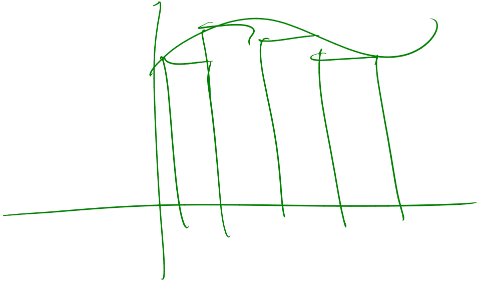
Theorem 3.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \mathrm{vol}(I)$ for all cuboids I.*



- **Goal:** Define $\int_X f \, d\mu$ (the Lebesgue integral).
- **Idea:**
  - ▷ Say $s : X \to \mathbb{R}$ is such that $s = \sum_1^N a_i \mathbf{1}_{A_i}$, for some $a_i \in \mathbb{R}$, $A_i \in \Sigma$. (Called *simple functions*.)
  - ▷ Define $\int_X s \, d\mu = \sum_1^N a_i \mu(A_i)$.
  - ▷ If $f \geqslant 0$, define $\int_X f \, d\mu = \sup_{s \leqslant f} \int_X s \, d\mu$.
- Will do this after constructing the Lebesgue measure.

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$





(Riemann)

4. Construction of the Lebesgue Measure

4.1. Lebesgue Outer Measure.

$[a,b]$ (a,b), $(a,b]$. $[a,b)$

Definition 4.1. We say $I \subseteq \mathbb{R}$ is a *cell* if I is a finite interval. Define $\ell(I) = \sup I - \inf I$.

Definition 4.2. We say $I \subseteq \mathbb{R}^d$ is a *cell* if it is a product of cells. If $I = I_1 \times \cdots \times I_d$, then define $\ell(I) = \prod_{i=1}^d \ell(I_i)$.

Remark 4.3. $\ell(I) = \ell(\mathring{I}) = \ell(\bar{I})$.

Remark 4.4. $\emptyset = \prod_1^d (a,a)$, and so $\ell(\emptyset) = 0$.

Remark 4.5. For all $\alpha \in \mathbb{R}^d$, $\ell(I) = \ell(I + \alpha)$.

Theorem 4.6. *There exists a (unique) measure λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\lambda(I) = \ell(I)$ for all cells I.*

Question 4.7. *How do you extend ℓ to other sets?*

Definition 4.8 (Lebesgue outer measure)**.** Given $A \subseteq \mathbb{R}^d$, define $\lambda^*(A) = \inf\left\{\sum_1^\infty \ell(I_k) \;\middle|\; A \subseteq \bigcup_1^\infty I_k, \text{where } I_k \text{ is a cell}\right\}$.

Remark 4.9. Some authors use m^* instead of λ^*.

Remark 4.10. λ^* is defined on $\mathcal{P}(\mathbb{R}^d)$; but only "well behaved" on a σ-algebra.

Question 4.11. *What is* $\lambda^*(\emptyset)$*? What is* $\lambda^*(\mathbb{R}^d)$*?*

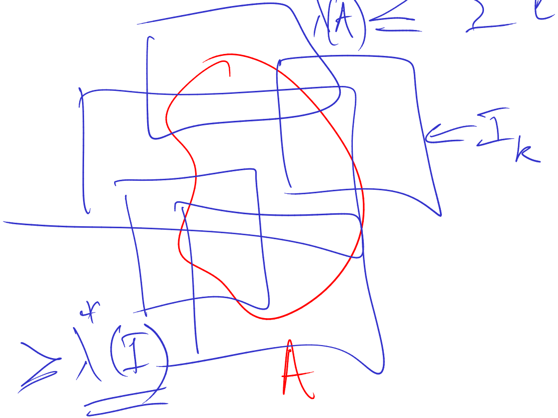
0

$+\infty$.

① If $A \subseteq B$, $\lambda^*(A) \le \lambda^*(B)$

② $\lambda^*(I) = \ell(I)$ (IOU)

$\lambda(A) \le \sum \ell(I_k)$



$\lambda^*(\mathbb{R}^d) \ge \underline{\lambda^*(I)}$

Proposition 4.12. *If $E \subseteq F$, then $\lambda^*(E) \leqslant \lambda^*(F)$.*

Proposition 4.13. *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then $\lambda^*(\cup_1^\infty E_i) \leqslant \sum_1^\infty \lambda^*(E_i)$.*

Pf: $\forall i, \ \forall \varepsilon > 0 \ \exists$ cells $I_{i,k} \ni \lambda^*(E_i) \geq \sum_k \ell(I_{i,k}) - \frac{\varepsilon}{2^i}$

$(E_i \subseteq \bigcup_k I_{i,k})$.

Clearly $\bigcup_{i,k} I_{i,k} \supseteq \bigcup_{i=1}^{\infty} E_i$

$\Rightarrow \lambda^*\left(\underline{\underline{\bigcup_{i=1}^{\infty} E_i}}\right) \leq \sum_{i,k} \ell(I_{i,k}) \leq \sum_{i=1}^{\infty}\left(\lambda^*(E_i) + \frac{\varepsilon}{2^i}\right) = \sum \lambda^*(E_i) + \varepsilon$

QED.

Proposition 4.14. *Let* $\underline{A}, \underline{B} \subseteq \mathbb{R}^d$*, and suppose* $\boxed{d(A,B) > 0}$*. Then* $\underline{\lambda^*(A \cup B) = \lambda * (A) + \lambda * (B)}$*.*

Proof: Only need to show $\underline{\lambda^*(A \cup B) \geqslant \lambda^*(A) + \lambda^*(B)}$. If $\underline{\lambda^*(A \cup B) = \infty}$, we are done, so assume $\underline{\lambda^*(A \cup B) < \infty}$.

Pick $\varepsilon > 0$.

$\exists$ colle $I_k \ni A \cup B \subseteq \bigcup_1^\infty I_k$ & $\lambda^*(A \cup B) \geqslant \sum l(I_k) - \varepsilon$

Let $\{I_k\} = \{J_k\} \cup \{J_k'\} \ni J_k \cap A \neq \phi$

(Subdivide I_k if necessary $\ni$ diam $(I_k) < d(A,B)$) $\longrightarrow$ & $J_k \cap B = \phi$ $\quad J_k' \cap B \neq \phi$ & $J_k' \cap A = \phi$

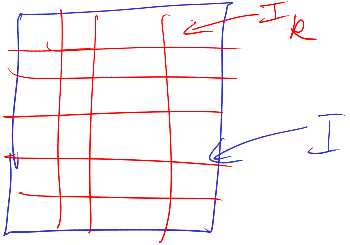
$$\left.\begin{array}{l} \lambda^*(A) \leq \sum l(J_k) \\ \lambda^*(B) \leq \sum l(J_k') \end{array}\right\} \lambda^*(A) + \lambda^*(B) = \sum l(J_k) + \sum l(J_k') = \sum l(I_k) \leq \underline{\lambda^*(A \cup B)} + \varepsilon$$

QED.

Proposition 4.15. *If $I \subseteq \mathbb{R}^d$ is a cell, then $\lambda^*(I) = \ell(I)$.*

Lemma 4.16. *If $\{I_k\}$ divide I by hyperplanes, then $\sum \ell(I_k) = \ell(I)$.* (Dist lect).

Lemma 4.17. $\lambda^*(A) = \inf\{\sum \ell(I_i) \mid A \subseteq \cup I_k,$ *and I_k are all* open *cells*$\}$.



Pf: $\varepsilon > 0$. $\exists \{I_j\} \ni A \subseteq \bigcup I_j$

& $\lambda^*(A) \geq \sum \ell(I_j) - \varepsilon$

Let $J_k \supseteq I_k$ & $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$ & J_k open.

$$\Rightarrow \underline{\sum \ell(J_k)} \leq \sum \ell(I_k) + \varepsilon \leq \lambda^*(A) + 2\varepsilon$$

QED.

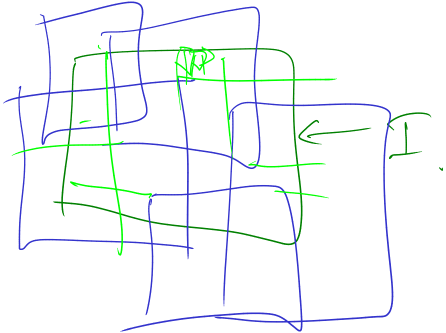
Proof of Proposition 4.15: Suppose first I is closed (hence compact). Pick $\varepsilon > 0$.

NTS $\ell(I) = \lambda^*(I)$. Know $\lambda^*(I) \le \ell(I)$. NTS $\lambda^*(I) \ge \ell(I)$.

Pick $\varepsilon > 0$. $\exists$ open cells I_k s.t. $I \subseteq \bigcup_1^\infty I_k$ & $\lambda^*(I) \ge \sum_1^\infty \ell(I_k) - \varepsilon$

Use finite subcover. $\lambda^*(I) \ge \sum_1^N \ell(I_k) - \varepsilon \ge \ell(I) - \varepsilon$. QED.

↑ extend faces of each cell to divide I by hyperplanes.



Proposition 4.18 (Translation invariance). *For all $A \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $\lambda^*(A) = \lambda^*(\alpha + A)$.*

Last time: $\lambda^* \longrightarrow$ Lebesgue OUTER measure. $\longrightarrow \lambda^*(A) = \inf\left\{ \sum_k \ell(I_k) \;\middle|\; \bigcup I_k \supseteq A \text{ \& } I_k\text{'s a d} \right\}$

$\hookrightarrow \lambda^*\left(\bigcup_{i}^{\infty} A_i\right) \leq \sum_{i}^{\infty} \lambda^*(A_i)$ (sub add)

Today: Want a meas out of λ^*

$\ell(I + \alpha) = \ell(I)$

"Haar Measure"

4.2. **Carathéodory Extension.** Our goal is to start with an *outer measure*, and restrict it to a *measure*.

Definition 4.19. We say μ^* is an outer measure on X if:

(1) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, and $\mu^*(\emptyset) = 0$.
(2) If $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$.
(3) If $A_i \subseteq X$ (not necessarily disjoint), then $\mu^*(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \mu^*(A_i)$. (countable sub-additivity)

Example 4.20. Any measure is an outer measure.

Example 4.21. The Lebesgue outer measure is an outer measure.

Theorem 4.22 (Carathéodory extension). *Let* $\Sigma \overset{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\ \forall A \subseteq X\}$. *Then* Σ *is a* σ*-algebra, and* μ^* *is a measure on* (X, Σ).

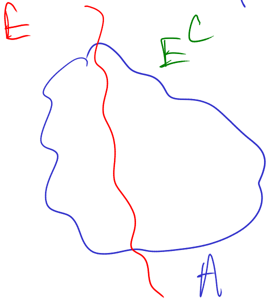
Remark 4.23. Clearly $\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \geqslant \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c. If E is nice, hopefully the overlap is small.

① $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$ ∀ A, E (sub add).

② For nice sets E ($E \in \Sigma$) also have $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

↳ λ^*; A ← cover by cells: $A \subseteq \bigcup_1^\infty I_k$: $\mu^*(A) \geq \sum \ell(I_k) - \varepsilon$.



$\{I_k\} = \{J_k\} \cup \{J'_k\}, \cup \{J''_k\}$

$J_k \cap E \neq \emptyset$ $\quad$ $J'_k \cap E^c \neq \emptyset$ $\quad$ $J''_k \cap E \neq \emptyset$ ←

$\cancel{J_k \cap E^c = \emptyset}$ $\quad$ $\cancel{J'_k \cap E = \emptyset}$ $\quad$ $J''_k \cap E^c \neq \emptyset$ ←

$$\mu^*(A) \geq \sum l(I_k) - \varepsilon = \sum l(J_k) + \sum l(J_k') - \sum l(J_k'') - \varepsilon$$

$$\geq \underline{\mu^*(A \cap E)} + \underline{\mu^*(A \cap E^c)} - \boxed{\sum l(J_k'')} - \underset{\sim}{\varepsilon}$$

Hope this is small!

Proof of Theorem 4.22 (Carathedory)

(1) $\emptyset \in \Sigma$. ✓

(2) $E \in \Sigma \implies E^c \in \Sigma$. ✓

(3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \dots, E_n \in \Sigma \implies \cup_1^n E_i \in \Sigma$.)

NTS $\mu^*(A) = \underbrace{\mu^*(A \cap \emptyset)}_{0} + \underbrace{\mu^*(A \cap X)}_{\mu^*(A)}$

$E \in \Sigma \implies \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$

$E^c \in \Sigma$? $\mu^*(A \cap E^c) + \mu^*(A \cap \underbrace{(E^c)^c}) = \mu^*(A)$ ✓

③ NTS $E \cup F \in \Sigma$.

NTS $\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c) = \mu^*(A)$ $\forall A$

LHS

$= \mu^*(A \cap (E \cup F) \cap E) + \mu^*(A \cap (E \cup F) \cap E^c) + \mu^*(A \cap (E \cup F)^c)$ $(\because E \in \Sigma)$.

$= \mu^*(A \cap E) + \underbrace{\mu^*(A \cap F \cap E^c) + \mu^*(A \cap E^c \cap F^c)}_{\mu^*(A \cap E^c)}$ $(\because F \in \Sigma)$

$= \mu^*(A)$ QED

(4) If $E_1, \dots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\boxed{\mu^*(A \cap (\cup_1^n E_i)) = \sum_1^n \mu^*(A \cap E_i).}$

NTS $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F)$ $\left(\forall A \subseteq X,\ E, F \subseteq \Sigma \right)$
$E \cap F = \phi$

Pf: $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap (E \cup F) \cap E) + \mu^*(\_\_\_\_\_\_ \cap E^c)$

$= \mu^*(A \cap E) + \mu^*(A \cap F)$

QED.

(5) Σ is closed under countable *disjoint* unions, and μ^* is countably additive on Σ.

($\Rightarrow$ Σ is a σ-alg) $\Rightarrow$ $\mu^*|_\Sigma$ is a meas!

Proof: Let $E_1, E_2, \dots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

NTS $\bigcup_1^\infty E_i \in \Sigma$. $\Leftrightarrow$ NTS $\forall A$, $\mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c) = \mu^*(A)$.

$$\mu^*(A) = \mu^*(A \cap (\bigcup_1^N E_i)) + \mu^*(A \cap (\bigcup_1^N E_i)^c) \qquad (\because \bigcup_1^N E_i \in \Sigma)$$

$$\overset{\geq}{\cancel{=}} \sum^N \mu^*(A \cap E_i) + \mu^*(A \cap (\bigcup_1^{\cancel{N}\,\infty} E_i)^c)$$

$\forall N$ send $N \to \infty$

$$\Rightarrow \mu^*(A) \geq \sum_1^\infty \mu^*(A \cap E_i) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c)$$

$$\geq \mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c) \geq \mu^*(A)$$

$$\Rightarrow \mu^*(A) = \mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c)$$

$\Rightarrow \bigcup_1^\infty E_i \in \Sigma$. QED.

& $\mu^*(A \cap (\bigcup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$

Remark 4.24. Note, the above shows $\boxed{\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i).}$

Definition 4.25. Define the *Lebesgue σ-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \cap \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 4.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

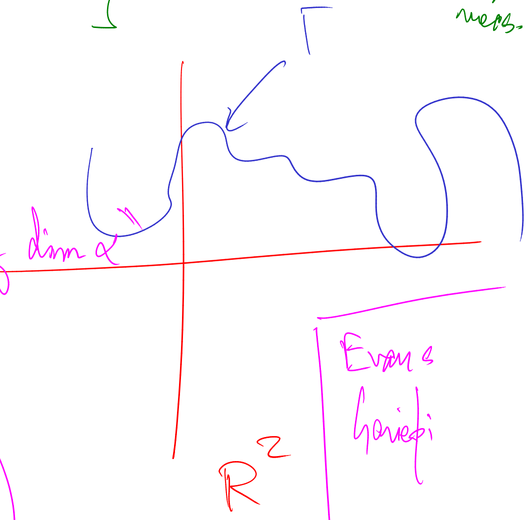
Remark 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$.

Question 4.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

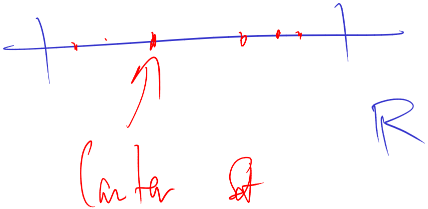
$\lambda^*(A) = \inf\{\{ \sum_i \pi r_i^{\cancel{2}\,\cancel{\alpha}\,\alpha} \mid \bigcup_{i=1}^{\infty} B(x_i, r_i) \supseteq A \}$ ← should be = lebesgue meas.

$\lambda^* =$ lebesgue meas on $\mathbb{R}^2$. $(\alpha > 0)$ → "Hausdorff measure of dim α

Q: $\lambda^*(\Gamma) = 0$



$0 = \lambda^*(C)$ →



Evans
Gariepi

Definition 4.25. Define the *Lebesgue σ-algebra* by $\boxed{\mathcal{L}(\mathbb{R}^d)} = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \overset{+}{\cancel{\cap}} \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 4.26. Define the *Lebesgue measure* by $\underline{\lambda(E) = \lambda^*(E)}$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

Remark 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$.

Question 4.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

Yes

↓

Proposition 4.29. *If $I \subseteq \mathbb{R}^d$ is a cell, then $I \in \mathcal{L}(\mathbb{R}^d)$.* (i.e. $\forall A$, $\lambda^*(A) = \lambda^*(A \cap I) + \lambda^*(A \cap I^c)$)

Proof: Only WTS $\lambda^*(A) \geq \lambda^*(A \cap I) + \lambda^*(A \cap I^c)$

Pick $\varepsilon > 0$. Pick a cell $J_\varepsilon \subseteq I$ + $d(J_\varepsilon, I^c) > 0$ and $\lambda^*(I - J_\varepsilon) < \varepsilon$

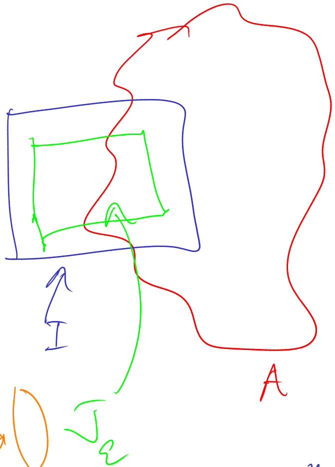
$$\Rightarrow \lambda^*(A) \geq \lambda^*(A \cap (J_\varepsilon \cup I^c)) \overset{\text{(sep add)}}{=} \underbrace{\lambda^*(A \cap J_\varepsilon)}_{\geq \lambda^*(A \cap I) - \varepsilon} + \lambda^*(A \cap I^c)$$

~~Hope~~

$$\lambda^*(A \cap I) \leq \lambda^*(A \cap J_\varepsilon) + \lambda^*(A \cap (I - J_\varepsilon)) \quad \text{(sub add)}$$

$$\leq \quad '' \quad + \lambda^*(I - J_\varepsilon)$$

$$\leq \quad ' \quad + \varepsilon \Rightarrow$$



$$\therefore \lambda^*(A) \geq \lambda^*(A \cap I) - \varepsilon + \lambda^*(A \cap I^c)$$

QED.

Proposition 4.30. $\boxed{\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d).}$

Null sets

Remark 4.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Pf: Any open set is a ctable union of cells

$\Rightarrow \mathcal{L}(\mathbb{R}^d) \supseteq$ all open sets

$\Rightarrow \mathcal{L}(\mathbb{R}^d) \supseteq \sigma(\quad\quad) = \mathcal{B}(\mathbb{R}^d)$ QED.

$N \in \mathcal{N}$. $(\lambda^*(N) = 0)$ Q: $N \in \mathcal{L}(\mathbb{R}^d)$? ✓

NTS

$$\lambda^*(A) \geq \underbrace{\lambda^*(A \cap N)}_{0} + \underbrace{\lambda^*(A \cap N^c)}_{\leq \lambda^*(A)} \leftarrow \text{True}$$

Here are two results that will be proved later:

Theorem 4.32. $\boxed{\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d).}$ *(In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.)*

Theorem 4.33. $\boxed{\mathcal{L}(\mathbb{R}^d) \subsetneq \mathcal{P}(\mathbb{R}^d).}$

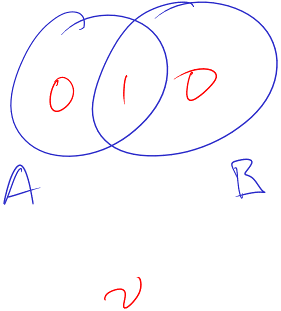
(Proofs later)

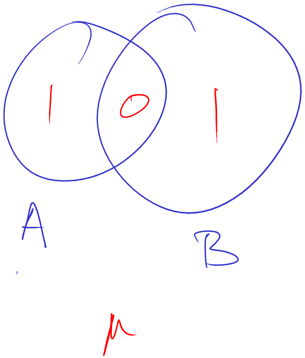
Theorem 4.34 (Uniqueness). *If μ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.*

Question 4.35. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose μ, ν are two measures which agree on $\mathcal{E}$. Must they agree on $\sigma(\mathcal{E})$?* $\sigma(\mathcal{E})$

$\mathcal{E} = \{A, B\}$

NO! ✓





Pf of 4.34: Know $\mu(I) = \lambda(I)\ \forall$ cells I.

Claim: $\mu \leq \lambda$

$A \in \mathcal{B}(\mathbb{R}^d)$. $A \subseteq \bigcup_1^\infty I_k \Rightarrow \mu(A) \leq \sum_1^\infty \mu(I_k) = \sum_1^\infty \lambda(I_k)$

$\Rightarrow \mu(A) \leq \inf\left\{ \sum_1^\infty \ell(I_k) \,\middle|\, \bigcup_1^\infty I_k \supseteq A \right\} = \lambda(A) \Rightarrow$ Claim.

Claim 2: Say E is <u>bdd.</u> Then $\lambda(E) \leq \mu(E)$

Pf: Find a cell I + $I \supseteq E$. $\mu(I-E) \leq \lambda(I-E) = \cancel{\lambda(I)} - \lambda(E)$

$\cancel{\mu(I)} - \mu(E)$ $\Rightarrow \mu(E) \geq \lambda(E)$ QED.

Claim 3: $\forall E,\ \lambda(E) \leq \mu(E)$

Pf: Write $E = \bigcup_1^\infty E_i$, E_i are disj & bdd & use claim 2.

QED.

5. Abstract measures

5.1. Dynkin systems.

Question 5.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?* NO.

▷ Clearly need Π to be closed under intersections.

Very minimum needed of an Π is

$$\underbrace{\bigcup_{1}^{\infty} \left(B(0,n) - B(0,n-1) \right) \cap E}_{E_i}$$

Question 5.2. *Let* $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. *Must* Λ *be a* σ*-algebra?*

▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$? NO

▷ If $A \subseteq B$, $A, B \in \Lambda$, must $B - A \in \Lambda$? (μ, ν, finite, Yes)

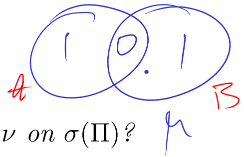
▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

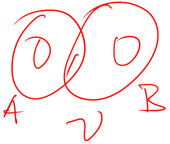
$$\mu(B-A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B-A)$$

5. Abstract measures

5.1. Dynkin systems.

Question 5.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*





▷ Clearly need Π to be closed under intersections.

Thm: $\Pi \rightarrow$ closed under int

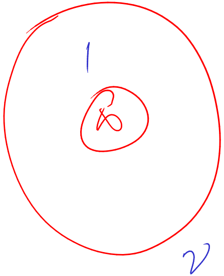
$\mu, \nu \rightarrow$ 2 finite measures

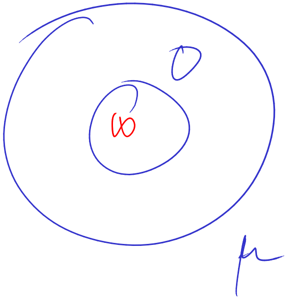
$\mu = \nu$ on Π

$\Rightarrow \mu = \nu$ on $\sigma(\Pi)$

$A \subseteq B$

$$\left.\begin{array}{l}\mu(A) = \nu(A)\\ \mu(B) = \nu(B)\end{array}\right\} \rightarrow \text{Must } \mu(B-A) = \nu(B-A)\,?$$





Question 5.2. *Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ-algebra?*

$\boxed{(\mu, \nu \text{ finite})}$

- ▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$?
- ▷ If $A \subseteq B$, $A, B \in \Lambda$, must $\boxed{B - A \in \Lambda?}$
- ▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

NO → Yes if ① $A \cap B = \phi$

② $A \subseteq B$

~~③ $\mu(A \cap B) = \nu(A \cap B)$~~

Yes. $\mu(B - A) = \mu(B) - \mu(A)$

$= \nu(B) - \nu(A)$

$$\mu\left(\bigcup_1^\infty A_i\right) = \lim_{i \to \infty} \mu(A_i) = \lim_{i \to \infty} \nu(A_i) = li\, \nu\left(\bigcup_1^\infty A_i\right)$$

Yes

Definition 5.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if:

(1) $X \in \Lambda$

(2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$.

(3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^\infty A_n \in \Lambda$.

Definition 5.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π-system if whenever $A, B \in \Pi$, we have $A \cap B \in \Pi$.

Lemma 5.5 (Dynkin system lemma). *If Π is a π-system, and $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \sigma(\Pi)$.*

Corollary 5.6. *If μ, ν are finite measures such that $\mu = \nu$ on Π, and Π is closed under intersections, then $\mu = \nu$ on $\sigma(\Pi)$.* (and assume $X \in \Pi$).

Pf: $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$.

Λ is a λ-system (prev slide)

$\Lambda \supseteq \Pi$ (assumption).

Dynkin $\implies \Lambda \supseteq \sigma(\Pi)$

QED.

Proof of Lemma 5.5

(1) The arbitrary intersection of λ-systems is a λ-system. $\boxed{\text{So it make sense to talk about } \lambda(\Pi).}$

(2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.

(3) If Λ is *both* a π-system and a λ-system, then Λ is a σ-algebra.

($\lambda(\Pi) = \lambda$-system gen by Π).

Pf of ③: NTS → If $A, B \in \Lambda$ then $A \cup B \in \Lambda$.

Pf: $A \cap B \in \Lambda$ (Λ is a π-sys)

$A - (A \cap B) \in \Lambda$ → (Λ is a λ-sys)

$B - (A \cap B) \in \Lambda$ ↗

$(X - (\underline{A - B})) - B$ ✓ Complex

$A, B \in \Lambda \Rightarrow A^c, B^c \in \Lambda$

$\Rightarrow A^c \cap B^c \in \Lambda$ (π-sys)

$\Rightarrow A \cup B = (A^c \cap B^c)^c \in \Lambda$ QED.

Since Λ is closed under inc ctable unions

$\Rightarrow \Lambda$ is a σ-alg.

(4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.

(5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. Then Λ_C is a λ-system.

Pf: ① $X \in \Lambda_C$ (Yes: $X \cap C \in \lambda(\Pi)$? ← Yes.)

② $A, B \in \Lambda_C$, $A \subseteq B$. NTS $B - A \in \Lambda_C$.

i.e. NTS $(B-A) \cap C \in \lambda(\Pi)$ ✓

$$(B-A) \cap C = \underbrace{\underset{\lambda(\Pi)}{\underset{\cap}{(B \cap C)}} - \underset{\lambda(\Pi)}{\underset{\cap}{(A \cap C)}}}_{\substack{\cap \\ \lambda(\Pi)}}$$

③ Inc unions → True (check).

(6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.

① ▷ Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.

▷ For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

Pf: $\Lambda_D \supseteq \Pi$ (Π is a π-system)

Know Λ_D is a λ-sys (step ⑤). $\Rightarrow \Lambda_D \supseteq \lambda(\Pi)$. QED.

$\Rightarrow \forall B \in \lambda(\Pi)$, $B \in \Lambda_D \Rightarrow B \cap D \in \lambda(\Pi)$.

Pf: $\Lambda_B \rightarrow$ is a λ-system (step 5)

$\Lambda_B \supseteq \Pi$ ✓

$\forall D \in \Pi$ must $D \cap B \in \lambda(\Pi)$?

Yes by

$\Rightarrow \Lambda_B \supseteq \lambda(\Pi)$.

$\Rightarrow \forall C \in \lambda(\Pi)$, $C \in \Lambda_B \Rightarrow B \cap C \in \lambda(\Pi)$

QED.

5.2. Regularity of measures.

→ μ is a measure $(X, \mathcal{B}(X))$.

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(Radon) (1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U$ is compact$\}$. ← (inner regular)
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U$ open$\}$. ← (outer regular).

Motivation:

- ▷ Approximation of measurable functions by continuous functions
- ▷ Differentiation of measures
- ▷ Uniqueness in the Riesz representation theorem

Question 5.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K$ compact$\}$ for all Borel sets A?* (False in general.)

$\inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U)$

$\sup_{K \subseteq U} \mu(K)$

$(K \not\subseteq A)$

True when $X = \mathbb{R}^d$ (closed sets)

Thm: X cpt & μ finite then μ is regular

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is regular, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\underline{\underline{\mu(U - C) < \varepsilon}}$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. □

Theorem 5.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. Then μ is regular. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ closed such that $\mu(U - K) < \varepsilon$.*

5.2. **Regularity of measures.**

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$.
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$.

Motivation:

- Approximation of measurable functions by continuous functions
- Differentiation of measures
- Uniqueness in the Riesz representation theorem

Question 5.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?*

(False → find a c-ex)

satisfies ① above. ⇒ μ is regular &

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is ~~regular~~, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\boxed{\mu(U - C) < \varepsilon}$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. □

Theorem 5.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. $\boxed{\text{Then } \mu \text{ is regular}}$. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ ~~closed~~ cpt such that $\mu(U - K) < \varepsilon$.*

$(\text{Leb } \mu = \lambda)$

$\exists$ K cpt & U open s.t.

$K \subseteq A \subseteq U$

& $\mu(U - K) < \varepsilon$

$\mu(A) = \sup_{K \subseteq A} \mu(K)$ (K cpt)

$\nexists$ A

$\forall \varepsilon > 0$, $A \in \mathcal{B}(X)$ $\exists$ $U \supseteq A$, U open, $K \subseteq A$, K cpt $\Big\}$ $\mu(U - K) < \varepsilon$.

Proof:

(1) Let $\Lambda = \{A \in \mathcal{B}(X) \mid \forall \varepsilon > 0,\ \exists K \subseteq A$ compact, $U \supseteq A$ open, such that $\mu(U - K) < \varepsilon\}$.

(2) Λ contains all open sets.

Let $U \subseteq X$ open. NTS $\forall \varepsilon > 0\ \exists K \subseteq U$ cpt s.t. $\mu(U - K) < \varepsilon$.

Write $U = \bigcup_{1}^{\infty} K_n$, $K_n \subseteq X$ is cpt & $K_n \subseteq K_{n+1}$

$(\text{Eg } K_n = \{x \in X \mid d(x, U^c) \geq \frac{1}{n}\})$

$\Rightarrow \mu(U) = \lim_{n \to \infty} \mu(K_n)$ QED

(3) Λ is a λ-system. (In this case it's easy to directly show that Λ is a σ-algebra.)

(4) Dynkin's Lemma implies $\Lambda \supseteq \mathcal{B}(X)$, finishing the proof.

Pf of ③: ① $X \in \Lambda$ (∵ X is open).

NTS: ② $A_1, A_2 \in \Lambda$, $A_1 \subseteq A_2 \Rightarrow A_2 - A_1 \in \Lambda$

Pf: Pick $\varepsilon > 0$, $\exists K_i, U_i \ni K_i$ cpt, U_i open $K_i \subseteq A_i \subseteq U_i$

& $\mu(U_i - K_i) < \varepsilon$ (∵ $A_i \in \Lambda$)

$$K_2 - U_1 \subseteq A_2 - A_1 \subseteq \underbrace{U_2 - K_1}_{\text{open}}, \quad \& \ \mu((U_2 - K_1) - (K_2 - U_1)) \le \mu(U_2 - K_2) + \mu(U_1 - K_1) = 2\varepsilon$$

QED

③ $A_1, A_2 \cdots \in \Lambda$. NTS $\bigcup_1^\infty A_i \in \Lambda$.

$A_i \subseteq A_{i+1}$. Pf: $B_i = A_i - A_{i-1}$. Then $\bigcup A_i = \bigcup_{\text{disj}} B_i$

$\forall i,\ B_i \in \Lambda \Rightarrow \exists K_i$ cpt & U_i open $\ni$ $K_i \subseteq B_i \subseteq U_i$

& $\mu(U_i - K_i) < \frac{\varepsilon}{2^i}$

Certainly $\underbrace{\bigcup_{i}^{\infty} K_i}_{\text{not cpt}} \subseteq \bigcup A_i \subseteq \underbrace{\bigcup_{i}^{\infty} U_i}_{\text{open}}$ & $\underbrace{\mu\left(\bigcup U_i - \bigcup K_i\right) \leq \sum \mu(U_i - K_i)}$ $< \varepsilon$.

✓ ✓

take $\bigcup_{i}^{N}$ for some large N & finish.

Corollary 5.11. *Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subset X$ such that $\bar{B}_n \subset \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\mu(B_n) < \infty$. Then μ is regular. Further:*

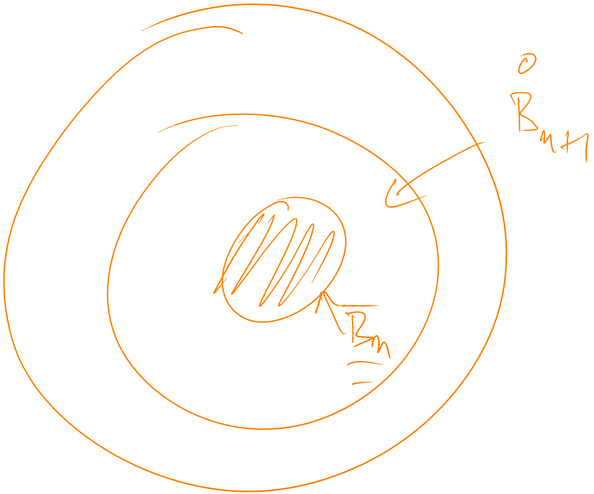
(1) For any Borel set A, $\mu(A) = \sup\{\mu(K) \mid K \subseteq K$ is compact$\}$.

(2) For any $\varepsilon > 0$, there exists $U \supseteq A$ open and $C \subseteq A$ closed such that $\mu(U - C) < \varepsilon$.

Proof. On homework. ☐

$$X = \bigcup_{1}^{\infty} B_n$$

(Eg: $X = \mathbb{R}^d$. $B_n = B(0, n)$)



Theorem 5.12. *Let* $A \in \mathcal{L}(\mathbb{R}^d)$, $\lambda(A)$. (λ = Lebesgue meas).

(1) $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\} = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ compact}\}$.

(2) *There exists* $\varepsilon > 0$, $C \subseteq A$ *closed and* $U \supseteq A$ *open such that* $\lambda(U - C) < \varepsilon$.

Pf of ①: $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\}$ ✓

($\because$ $\lambda(A) = \lambda^*(A) = \inf\{\sum l(I_k) \mid \bigcup I_k \supseteq A,\ I_k \text{ open cells}\}$

$= \inf\{\lambda(\bigcup I_k) \mid$ 〃 〃 $\}$ QED)

NTS $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ cpt}\}$.

↳ Case 1: A bdd. $\lambda(A) = \lambda(I) - \lambda(I - A)$, $I \supseteq A$ is some closed cell

& use $\lambda(I - A) = \inf_{U \supseteq I - A} \lambda(U)$.

Case 2: $A = \bigcup_{1}^{\infty} A \cap (B(0, n+1) - B(0, n))$.

$\Longrightarrow$ ①

Pf of ②: Case 1: A bdd $\longrightarrow$ Use part 1.

Case 2: Write $A = \bigcup_1^\infty \underbrace{A \cap \left(B(0,n+1) - B(0,n)\right)}_{A_n}$.

Case 1 $\Rightarrow \exists\, U_n \supseteq A_n$ & $K_n \subseteq A_n \ni U_n$ open, K_n cpt & $\mu(U_n - K_n) < \frac{\varepsilon}{2^n}$

Let $U = \bigcup_1^\infty U_n \longleftarrow$ open.

& $C = \bigcup_1^\infty K_n \longleftarrow$ closed

$\Big\} \longrightarrow$ done!

5.3. **Non-measurable sets.**

Theorem 5.13. *There exists* $E \subseteq \mathbb{R}$ *such that* $E \notin \mathcal{L}(R)$.

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)
(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.
(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.
(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.
(5) $\lambda(E) > 0$

HW Q 3/4:

Hausdorff meas: $A \subseteq \bigcup_{i}^{\infty} E_i$, $\text{diam}(E_i) < \delta$

$\lim_{\delta \to 0} H_{\alpha,\delta}(A) = H_\alpha(A) \leftarrow H_{\alpha,\delta}(A) = \inf\{ \sum c_\alpha \, \text{diam}(E_i)^\alpha \mid \}$

Sph meas: $S_{\alpha,\delta}(A) = \inf\{ \sum c_\alpha \, \text{diam}(B(x_i, r_i))^\alpha \mid A \subseteq \bigcup_{i}^{\infty} B(x_i, r_i) \; \& \; 2r_i \le \delta \}$

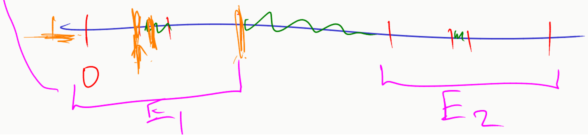
$S_\alpha \overset{?}{=} H_\alpha \Rightarrow H_\alpha \le S_\alpha$ $\qquad S_\alpha \le 2^d H_\alpha$

Claim: $\boxed{H_\alpha \ne S_\alpha}$ in general

Eg: Cantor Set.

$\delta = 1/3$ cover C by ① sets of diam $1/3$

② balls of diam $1/3$



Is E_1 a Ball in the Cantor set (NO)

5.3. Non-measurable sets.

Theorem 5.13. *There exists* $E \subseteq \mathbb{R}$ *such that* $E \notin \mathcal{L}(R)$.

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)

(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.

(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.

(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.

(5) $\lambda(E) > 0$

(axiom of choice).

$\forall \alpha \in \mathbb{R}$, $E \cap C_\alpha = \{e_\alpha\}$.

Claim $E \notin \mathcal{L}(\mathbb{R})$.

Pf: By contradiction

Pf of ⑤:

$$\bigcup_{q \in \mathbb{Q}} E + q = \mathbb{R}$$

ctable disjt union

∞ meas.

$$\lambda(q + E) = \lambda(E) \ \forall q$$

$$\Rightarrow \lambda(E) > 0$$

QED.

(6) $\underline{\lambda(E) = 0 \text{ (contradiction).}}$

Let $n \in \mathbb{N}$. $E_n = E \cap (-n, n)$. $\lambda(E) = \lim_{n \to \infty} \lambda(E_n)$

Claim $\lambda(E_n) = 0 \quad \forall n$. ($\Rightarrow \lambda(E) = 0 \Rightarrow$ QED).

Pf of Claim: Let $A = \bigcup_{\substack{|q| \leq 1 \\ q \in \mathbb{Q}}} (q + E_n) \longleftarrow$ ctable disj union.

A is bdd $\Rightarrow \underline{\underline{\lambda(A) < \infty}}$ & $\lambda(q + E_n) = \lambda(E_n) \quad \forall q$.

$\Rightarrow \lambda(A) = \lambda(E_n) = 0$ QED.

Theorem 5.14. *Let* $\boxed{A \subseteq \mathbb{R}^d}$. *Every subset of* A *is Lebesgue measurable if and only if* $\boxed{\lambda(A^*) = 0}$.

Proof. One direction is immediate. The other direction is accessible with what we know so far, but we won't do the proof in the interest of time. □

Thm: $\exists A \subseteq \mathbb{R} \ni E \subseteq A,\ E \in \mathcal{L}(\mathbb{R}) \Rightarrow \lambda(E) = 0$

and $E \subseteq A^c,\ E \in \mathcal{L}(\mathbb{R}) \Rightarrow \lambda(E) = 0$

(IOU: $\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d)$

~~Easy~~ Easier when $d \geq 2$ ⟶ on HW)

5.4. Completion of measures.

Theorem 5.15. *$A \in \mathcal{L}(\mathbb{R}^d)$ if and only if there exist $F, G \in \mathcal{B}(\mathbb{R}^d)$ such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$.*

Pf: $\forall n \in \mathbb{N}$, $\exists$ U_n open, C_n closed $\ni$ $C_n \subseteq A \subseteq U_n$ & $\lambda(U_n - C_n) < \frac{1}{n}$

Let $F = \bigcup C_n$ (F_σ)

$G = \bigcap U_n$ (G_δ).

Clearly $F \subseteq A \subseteq G$ & $\lambda(G - F) \leq \lambda(C_n - U_n) \leq \frac{1}{n}$ $\forall n$

$\Rightarrow \lambda(G - F) = 0$ QED.

Corollary 5.16. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 5.17. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\rightarrow \Sigma_\mu \overset{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$. $= \mu(G)$ (IOU Thm well defined)

Done

Definition 5.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Theorem 5.19. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

$(\mathbb{R}^d, \mathcal{L}, \lambda)$ is comp

(IOU) $\Rightarrow (\mathbb{R}^d, \mathcal{B}, \lambda)$ is not

Check $\bar{\mu}$ is well defined:

Say $A \in \Sigma_\mu$ $\exists F_1 \subseteq A \subseteq G_1$ $F_i, G_i \in \Sigma$ & $\mu(G_i - F_i) = 0$

$F_2 \subseteq F \subseteq G_2$

$\Rightarrow F_1 \subseteq F_1 \cup F_2 \subseteq A \subseteq G_1 \cap G_2 \subseteq G_1$

$\mu(G_1 - F_1) = 0 \Rightarrow \mu((F_1 \cup F_2) - F_1) = 0 \Rightarrow \mu(F_2) = \mu(F_1) \Rightarrow \bar{\mu}$ well def

QED.

Corollary 5.16. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 5.17. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\Sigma_\mu \overset{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$.

Definition 5.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Theorem 5.19. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

Next time $\bar{\mu} \rightarrow$ well def. ① Σ_μ is a σ-alg. Pf: ⓐ $X \in \Sigma_\mu$.

ⓑ If $A \in \Sigma_\mu \Rightarrow A^c \in \Sigma_\mu$ (Pf: Find $F \subseteq A \subseteq G \Rightarrow G^c \subseteq A^c \subseteq F^c$)

ⓒ $A_i \in \Sigma_\mu$. NTS $\bigcup_i^\infty A_i \in \Sigma_\mu$ (Pf: $\forall i$ $\exists F_i \subseteq A_i \subseteq G_i$ + $\mu(G_i - F_i) = 0$
set $F = \cup F_i$, $G = \cup G_i$) ✓

② $\bar{\mu}$ a meas: $A_i \in \Sigma_\mu$, disj. Find F_i, G_i + $F_i \subseteq A_i \subseteq G_i$ & $\mu(G_i - F_i) = 0$

F_i disj $\cup F_i \subseteq \cup A_i \subseteq \cup G_i$ & $\mu(\cup G_i - \cup F_i) = 0 \Rightarrow \bar{\mu}(\cup A_i) = \mu(\cup F_i) = \sum \bar{\mu}(A_i)$ QED

Theorem 5.20. Σ_μ *is the smallest μ-complete σ-algebra containing Σ.*

Corollary 5.21. $\Sigma_\mu = \sigma(\Sigma \cup \mathcal{N})$.

Corollary 5.22. $\mathcal{L}(\mathbb{R}^d) = \sigma(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N})$.

Pf of 5.20: Say (X, τ, ν) is a meas space. $\tau \supseteq \Sigma$ & ν extends μ.

τ - complete $\Rightarrow \tau \supseteq \Sigma_\mu$.

Pf: $A \in \Sigma_\mu$. find $F, G \in \Sigma \ni F \subseteq A \subseteq G$ & $\mu(G-F)=0$

$\Rightarrow \nu(G-F)=0$

$\Rightarrow A-F$ is null $\left(A-F \subseteq G-F \longrightarrow \begin{matrix}\mu \text{ null} \\ \nu \text{ null}\end{matrix}\right)$

$A-F \in \tau \Rightarrow A \in \tau$ QED.

Remark 5.23. There could exist $\underline{\underline{\mu}}$-null sets that are not in Σ.

Say $\lambda \to$ lebesgue meas on $[0,1]$.

$\Rightarrow \Sigma = \{\phi, [0,1]\}$.

Q: Null sets of $(\underline{\underline{\Sigma}}, \lambda)$: $\mathcal{N} = \{\phi\}, \underset{\uparrow}{\cancel{\{0\}}}$

6. Measurable Functions

Definition 6.1. Let (X, Σ, μ) be a measurable space, and (Y, τ) a topological space. We say $f : X \to Y$ is measurable if $f^{-1}(\tau) \subseteq \Sigma$.

$\forall U \in \tau, f^{-1}(U) \in \Sigma$

Remark 6.2. Y is typically $[-\infty, \infty]$, $\mathbb{R}^d$, or some linear space.

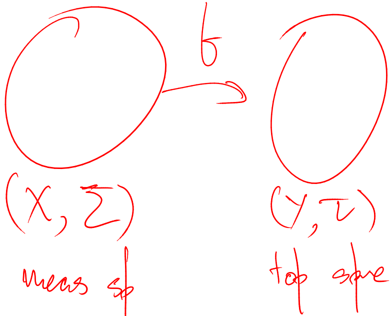
Remark 6.3. Any continuous function is Borel measurable, but not conversely.

Question 6.4. *Say $f : X \to Y$ is measurable. For every $B \in \mathcal{B}(Y)$, must $f^{-1}(B) \in \Sigma$?*

$\forall U \subseteq Y$ open, $f^{-1}(U) \in \Sigma$

Q: $\forall B \subseteq Y$ Borel, is $f^{-1}(B) \in \Sigma$?

Q2: $Y = \mathbb{R}$. $f : X \to Y$ is meas.



Q: $\forall B \in \mathcal{L}(\mathbb{R})$ is $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$?

$\boxed{\text{(NO!)}}$

Theorem 6.5. *Say $f\colon X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 6.6. *Let $f\colon X \to Y$ be arbitrary. Then $\Sigma' =$ ~~$\{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$~~ is a σ-algebra (on ~~Y~~ X).*

$\Sigma' = \{f^{-1}(B) \mid B \in \Sigma\}$

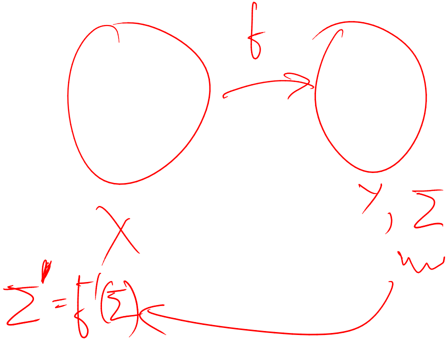
$\Sigma' = f^{-1}(\Sigma)$ (Σ is a σ-alg on Y)

Pf: $A_i \in f^{-1}(\Sigma) = \Sigma'$

Write $A_i = f^{-1}(B_i)$, $B_i \in \Sigma$

$f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i) = \bigcup A_i$

$\Rightarrow \bigcup A_i \in \Sigma'$. $\Rightarrow$ lemma



Pf of thm: $f^{-1}(\mathcal{B}(Y))$ is a σ-alg. f meas $\Rightarrow \forall\, U \subseteq Y$ open $\Rightarrow f^{-1}(U) \in \Sigma$.

$$\Rightarrow f^{-1}(\mathcal{B}(Y)) \subseteq \Sigma.$$

Corollary 6.7. *Let $f\colon X \to [-\infty, \infty]$. Then f is measurable if and only if for all $a \in \mathbb{R}$, we have $\boxed{\{f < a\}} \in \Sigma$.*

(X, Σ) meas space.

$$\{f<a\} = \{x \in X \mid f(x) < a\}$$

$$= f^{-1}([-\infty, a))$$

Pf: any open set can be expressed as a ct'ble union of intervals.

$$\& \quad f^{-1}(a,b) = \underbrace{\{f<b\}}_{\in \Sigma.} \cap \underbrace{\{f>a\}}_{\left(\bigcup_{n=1}^{\infty} \underbrace{\{f \geq a+\frac{1}{n}\}}_{\{f<a+\frac{1}{n}\}^c}\right) \in \Sigma} \in \Sigma$$

Lemma 6.8. *If $f\colon X \to \mathbb{R}^m$ is measurable, and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f\colon X \to \mathbb{R}^n$ is measurable.*

Question 6.9. *Is the above true if g was Lebesgue measurable?*

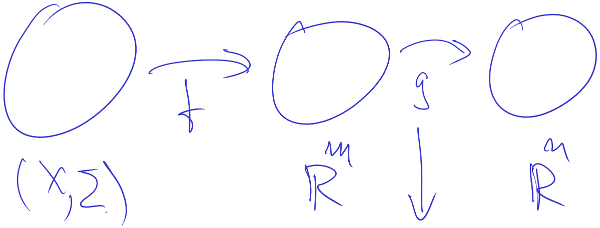
(False).

Pf Lemma: $U \subseteq \mathbb{R}^n$

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

$$g^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)$$

(By lemma,) $\Rightarrow f^{-1}(g^{-1}(U)) \in \Sigma$.



g is Borel meas

$(g^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)\ \forall U$ open$)$.

Theorem 6.5. *Say $f\colon X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 6.6. *Let $f\colon X \to Y$ be arbitrary, and Σ be a σ-algebra on X. Then $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$ is a σ-algebra (on Y).*

f – meas means $\forall\, U \subseteq Y$ open $\Rightarrow f^{-1}(U) \in \Sigma$.

($f\colon X \to Y$

Σ – σ-alg on X)

Say τ a σ-alg on Y. $f\colon X \to Y$

Then $f^{-1}(\tau)$ is a σ-alg on X (True $\to$ not useful)

(X, Σ) $f\colon X \to Y$. $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$

Σ' is a σ-alg.

If f meas $\Rightarrow \Sigma' \supseteq$ open sets $\Rightarrow \Sigma' \supseteq \mathcal{B}(Y)$ QED Thm

Corollary 6.7. *Let* $f\colon X \to [-\infty, \infty]$*. Then* f *is measurable if and only if for all* $a \in \mathbb{R}$*, we have* $\{f < a\} \in \Sigma$*.*

Lemma 6.8. *If $f\colon X \to \mathbb{R}^m$ is measurable, and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $\boxed{g} \underset{\Leftarrow}{\circ} f\colon X \to \mathbb{R}^n$ is measurable.*

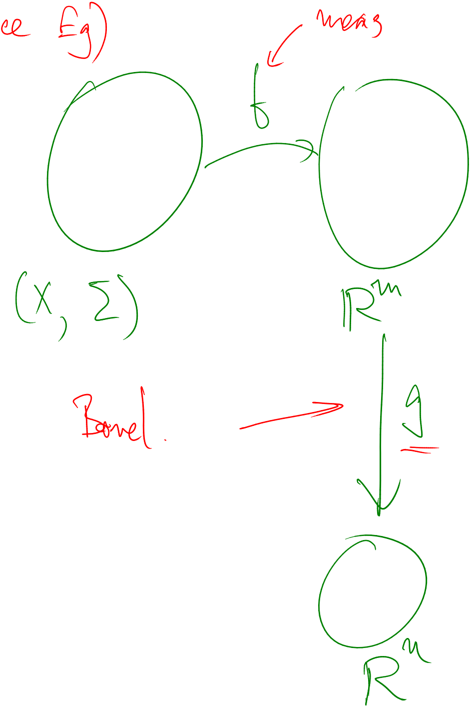
Question 6.9. *Is the above true if g was Lebesgue measurable?* (false IOU nice Eg)

Pf: $(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)})$

$= f^{-1}(\text{Borel})$ $(\because g \to \text{Borel})$

$\overset{\text{Thm}}{\underset{4}{=}} \in \Sigma$

QED.



Theorem 6.10. *Let* $f_n : X \to \mathbb{R}$ *be a sequence of measurable functions. Then* $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ *and* $\lim f_n$ *(if it exists) are all measurable.*

Q: $(f_n) \to f$ ptwise. f_n is R-int. Q: Is f R-int? (No)

Pf of thm: ① $\sup f_n$ meas: $\{\sup f_n \le \alpha\} = \bigcap_{n=1}^{\infty} \{f_n \le \alpha\}$

$(\{f<\alpha\} = \{x \mid f(x)<\alpha\})$

f_n meas $\forall n \Rightarrow \bigcap_{n=1}^{\infty} \{f_n \le \alpha\} \in \Sigma \Rightarrow \{\sup_n f_n < \alpha\} \in \Sigma$

By Lemma $\Rightarrow$ QED.

⑤ Let $f(x) = \begin{cases} \lim f_n(x) & \text{if the lim exists} \\ 0 & \text{otherwise.} \end{cases}$

Q: Is f meas. (each f_n is meas)

$E = \{ \underline{\limsup} f_n = \underline{\liminf} f_n \} \in \Sigma$ (check directly)

$\left(\mathbb{1}_E = \text{indicator fn of } E \right.$

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

$$f(x) = \lim_{n\to\infty} \underline{\mathbb{1}_E f_n}(x)$$

$$= \liminf_{n\to\infty} \underline{\underline{\mathbb{1}_E f_n}}$$

(check $\mathbb{1}_E f_n$ is meas $\forall n$)

$\Rightarrow f$ is meas

QED.

Theorem 6.11. *Let $\underline{f, g}\colon X \to \mathbb{R}$. The function $\underline{(f, g)}\colon X \to \mathbb{R}^2$ is measurable if and only if both $\underline{f}$ and $\underline{g}$ are measurable.*

$F = (f, g)$. $F : \underline{X} \to \mathbb{R}^2$. $F(x) = (f(x), g(x))$

Pf: ① Say f, g are meas. $\forall\, U, V \subseteq \mathbb{R}$, U, V open $\Rightarrow f^{-1}(U)$ & $g^{-1}(V) \in \Sigma$

$$F^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \in \Sigma.$$

Let $\Sigma' = \{E \subseteq \mathbb{R}^2 \mid F^{-1}(E) \in \Sigma\}$. Know Σ' is a σ-alg

Know $\Sigma' \supseteq \{\underline{U \times V} \mid U, V \subseteq \mathbb{R} \text{ open}\}$

$\Rightarrow \underline{\Sigma' \supseteq \mathcal{B}(\mathbb{R}^2)}$ QED

② Conversely: F meas. $\pi_1(x, y) = x$ cts fn ($\Rightarrow$ Borel)

$f(x) = \pi_1 \circ F(x) \Rightarrow f$ is meas (Borel composed with meas) QED

Corollary 6.12. *If* $\underline{f, g}\colon X \to \mathbb{R}$ *are measurable, then so is* $\underline{f+g}$, $\underline{fg}$ *and* $\underline{f/g}$ *(when defined).*

Pf: ① $f+g$: $F(x) = (f(x), g(x))$. $G(x,y) = x+y$

$$\left.f(x)+g(x) = \underbrace{G}_{\substack{\text{cts} \\ \text{(Borel)}}} \circ \underbrace{F}_{\text{meas}}(x)\right\} \Rightarrow \text{meas (by lemma)}$$

QED.

(Devils staircase)

Definition 6.13 (Cantor function)**.** Let C be the Cantor set, and $\boxed{\alpha = \log 2/\log 3}$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0,x])/H_\alpha(C)$.

(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C^c, and $f' = 0$ wherever defined.)

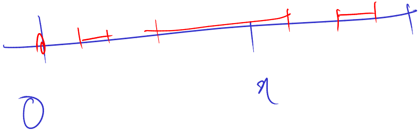
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)

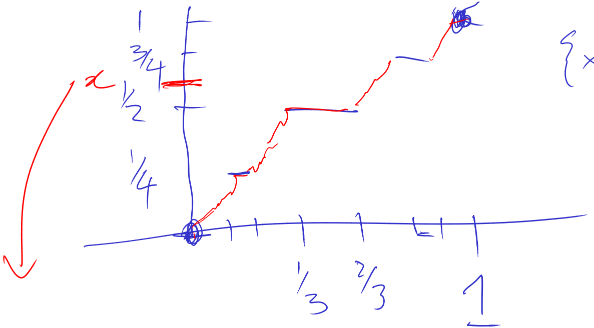
(3) Let $\boxed{g = f^{-1}}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$)).

$\leftarrow f(g(x)) = x$

Proposition 6.14. *The function $g\colon [0,1] \to C$ is a strictly injective Borel measurable function.*

$f(x) =$





$g(x) =$

$\{x \mid g(x) \le \alpha\}$

$= \{x \mid x \le f(\alpha)\}$

interval.

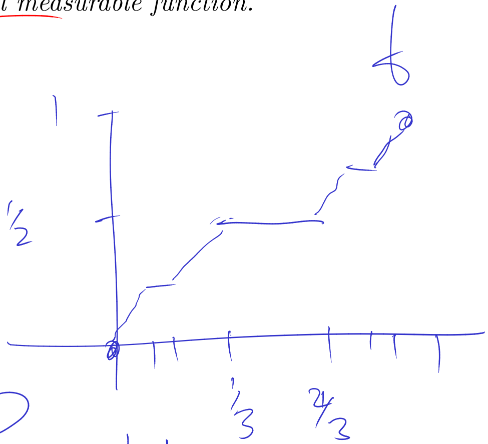
Definition 6.13 (Cantor function)**.** Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0,x])/H_\alpha(C)$.

(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C^{c}, and $f' = 0$ wherever defined.)
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)
(3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$).

Proposition 6.14. *The function $g\colon [0,1] \to C$ is a strictly injective Borel measurable function.*

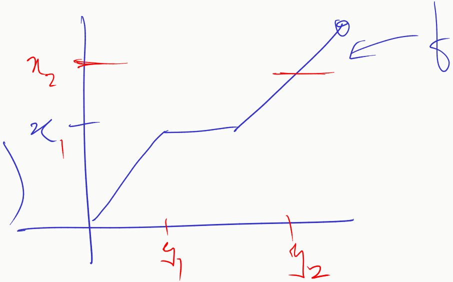
Pf f is cts:

$$f(x) - f(x - 1/n) = \frac{H_\alpha\left(\left(x - \frac{1}{n}, x\right] \cap C\right)}{H_\alpha(C)} \xrightarrow{n \to \infty} \frac{H_\alpha(\{x\} \cap C)}{H_\alpha(C)} = 0$$



$\Rightarrow f$ cts.

$g = \underline{f}^{-1}$: $\underline{g(x)} = \inf\{y \mid f(y) = x\}$



(Q: $\{y \mid f(y) = x\} \neq \emptyset$? $(x \in [0,1])$

A: Yes → int val thm)

f cts $\Rightarrow \inf\{y \mid f(y) = x\} = \min\{y \mid f(y) = x\}$

$\Rightarrow f(g(x)) = x$

Claim 1: g is <u>strictly</u> inc. ✓

Claim 2: g is Borel meas ($\because \{g < \alpha\}$ is an interval $\forall \alpha$)

Claim 3: Range$(g) \subseteq C$

Theorem 6.15. $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$.

Pf: Let $A \subseteq [0,1]$ be non meas

Q: $g(A) \rightarrow$ meas? Yes: $g(A) \subseteq \overset{\text{null}}{C} \Rightarrow g(A) \in \mathcal{L}(\mathbb{R})$

Q2: Is $g(A) \in \mathcal{B}(\mathbb{R})$?

NO! If $g(A) \in \mathcal{B} \Rightarrow \underbrace{g^{-1}(g(A))}_{A} \in \mathcal{B}(\mathbb{R})$ ($\because g$ is meas)

But $A \notin \mathcal{L}(\mathbb{R})$ by const. Contradiction

QED.

Theorem 6.16. *There exists $h_1, h_2\colon \mathbb{R} \to \mathbb{R}$ such that h_1 is $\mathcal{L}(\mathbb{R})$-measurable, h_2 is $\mathcal{B}(\mathbb{R})$ measurable, but $h_1 \circ h_2$ is not $\mathcal{L}(\mathbb{R})$ measurable.*

Remark 6.17. Note $h_2 \circ h_1$ has to be $\overset{\mathcal{L}}{\cancel{\mathcal{B}}}(\mathbb{R})$-measurable.

$E \subseteq X$,

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Pf: $A \subseteq [0,1]$, $A \notin \mathcal{L}(\mathbb{R})$

$g(A) \in \mathcal{L}(\mathbb{R})$

Let $h_1 = \mathbb{1}_{g(A)}$ (h_1 is $\mathcal{L}$- meas)

Let $h_2 = g$ (h_2 is $\mathcal{B}$ meas)

Note $h_1 \circ h_2 = \mathbb{1}_{g(A)} \circ g = \begin{cases} 1 & \text{on } A \\ 0 & \text{on } A^c \end{cases} = \mathbb{1}_A \leftarrow$ not $\mathcal{L}(\mathbb{R})$ meas

QED.

Definition 6.18. Let (X, Σ, μ) be a measure space. We say a property P holds almost everywhere if there exists a null set N such that P holds on N^c.

a.e.

Example 6.19. If f, g are two functions, we say $f = g$ almost everywhere if $\{f \neq g\}$ is a null set.

Example 6.20. Almost every real number is irrational.

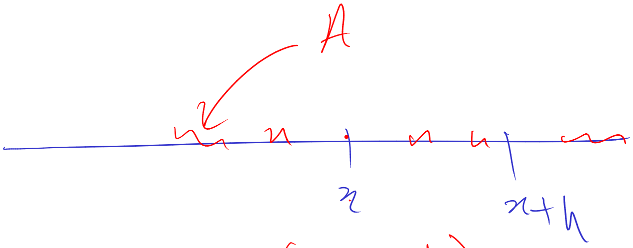
(IOU Pf)

Example 6.21. If $A \in \mathcal{L}(\mathbb{R})$, then $\lim_{h\to 0} \dfrac{\lambda(A \cap (x, x+h))}{h} = \mathbf{1}_A(x)$ for almost every x. (Contrast with HW3, Q3b)

Example 6.22. Let $x \in (0, 1)$, and p_n/q_n be the n^{th} convergent in the continued fraction expansion of x. Then $\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$.

$A = [0,1]$.

$$\mathbb{1}_A \overset{a.e}{=} \mathbb{1}_{[0,1)} = \left.\lim_{h\to 0} \frac{\lambda(A \cap (x, x+h))}{h}\right|$$



$$\left(\Rightarrow \nexists\, E \subseteq \mathbb{R} \text{ meas } \ni \forall\, (a,b),\ \frac{\lambda(E \cap (a,b))}{b-a} \in [k, 1-k] \right)$$

$x \in [0,1] \longrightarrow$ cont fraction for x

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Truncate to n terms. $\dfrac{p_n(x)}{q_n(x)}$ = nth conv of the

$$\lim_{n\to\infty} \frac{p_n(x)}{q_n(x)} = x \quad \forall x$$

Expect $q_n(x) \longrightarrow \infty$. Q: How fast?

$$\lim_{n\to\infty} \frac{\ln q_n(x)}{n} = \frac{\pi^2}{12\ln 2} \quad \forall x$$

almost every!

$\boxed{\text{Assume hereafter } (X, \Sigma, \mu) \text{ is complete.}}$

Proposition 6.23. *If $f = g$ almost everywhere and f is measurable, then so is g.*

Pf: NTS g meas. Let $N = \{f \neq g\}$ (null)

Pick $U \subseteq \mathbb{R}$ open.

$$g^{-1}(U) = \left(g^{-1}(U) \cap N^c\right) \cup \left(g^{-1}(U) \cap N\right)$$

$$= \underset{\in \Sigma}{\left(f^{-1}(U) \cap N^c\right)} \cup \underset{\in \Sigma}{\left(g^{-1}(U) \cap \underline{N}\right)} \Rightarrow \text{QED}.$$

Proposition 6.24. *If $(f_n) \to f$ almost everywhere, and each f_n is measurable, then so is f.*

Pf: $N^c = \{x \mid \lim_{n\to\infty} f_n(x) = f(x)\}$ $\quad$ N is null, ($\Rightarrow$ meas).

$$\mathbb{1}_{N^c} f = \underbrace{\lim_{n\to\infty} \underbrace{\mathbb{1}_{N^c} f_n}_{\text{meas}}}_{\text{meas (last time)}} \qquad (\forall x)$$

$\Rightarrow \mathbb{1}_{N^c} f$ is meas

$\mathbb{1}_{N^c} f = f$ a.e. $\Rightarrow f$ meas

QED

$\boxed{\text{~~Claim~~ HW 3b}}$ Claim $\nexists\, E \subseteq \mathbb{R} \ni \forall$ intervals I, $\dfrac{\lambda(E \cap I)}{\lambda(I)} \in [\kappa, 1-\kappa]$,

$(\kappa > 0)$

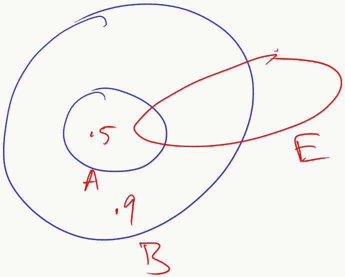
$$\Lambda = \{ A \in \mathcal{B} \mid \kappa \lambda(A) \le \lambda(A \cap E) \le (1-\kappa)\lambda(A) \}$$

$\kappa = .2$

① $\overset{[0,1]}{\cancel{\mathbb{R}}} \in \Lambda$

② $A \subseteq B$, $A, B \in \Lambda$, NTS $B - A \in \Lambda$

$\lambda($



Definition 6.25. A function $s\colon X \to \mathbb{R}$ is called *simple* if s is measurable, and has finite range (i.e. $s(\underset{X}{\cancel{\mathbb{R}}}) = \{a_1, \dots a_n\}$).

Question 6.26. *Why bother with simple functions?*

Eg: $A \in \Sigma$. $\quad s = \mathbb{1}_A$ $\quad \left(\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases} \right)$

$$\left(A \in \Sigma \Rightarrow s \text{ meas } \left(\because \{s < \alpha\} = \begin{cases} X & \alpha > 1 \\ A & \alpha \in (0,1) \leftarrow \in \Sigma \text{ (by ass.)} \\ \emptyset & \alpha \le 0 \end{cases} \right) \right).$$

$A_1, \dots A_n \in \Sigma$. $\quad a_1, \dots a_n \in \mathbb{R}$,

$$s = \sum_{1}^{n} \underline{a_i}\, \mathbb{1}_{\underline{A_i}}.$$

$\otimes$: $\displaystyle\int_X s \, d\mu \overset{\text{def}}{=} \sum a_i \mu(A_i) \leftarrow$

If f not simple define $\int f$ by approx f by simple fns.

Theorem 6.27. *If $f \geqslant 0$ is a measurable function, then there exists a sequence of simple functions (s_n) which increases to f.*

Corollary 6.28. *If $f\colon X \to \mathbb{R}$ is measurable, then there exists a sequence of simple functions (s_n) such that $(s_n) \to f$ pointwise, and $|s_n| \leqslant |f|$.* (Dominated).

Pf:

$$f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right) = A_{k,n} \in \Sigma.$$

Try $s_n = \sum_{k=0}^{n^2} \frac{k}{n} \mathbb{1}_{A_{k,n}}$ ($(s_n) \to f$ but need not be inc)

$$f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) = A_{k,n}.$$ & let $s_n = \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \mathbb{1}_{A_{k,n}}$.

s_n simple, $s_{n+1} - s_n \geqslant 0$

& $|s_n(x) - f(x)| \leq \frac{1}{2^n} \Rightarrow (s_n) \to f$ & (s_n) inc

QED.

Pf of Cor: $f^+ = \max\{f, 0\} = f \vee 0$ (meas)

$f^- = -\min\{f, 0\} = -(f \wedge 0)$ (meas)

$f = f^+ - f^-$.

By thm $\exists\ (s_n)$ simple + $(s_n) \longrightarrow f^+$
& $\exists\ (t_n)$ simple + $(t_n) \longrightarrow f^-$

then $(s_n - t_n) \longrightarrow f$, $s_n - t_n$ is simple
& $|s_n - t_n| \leq |f|$. QED.

Q: f meas $\not\Rightarrow$ f cts

Q: ? $\exists f$ meas + f is not cts anywhere?

Yes: $f = \mathbb{1}_{\mathbb{Q}}$.

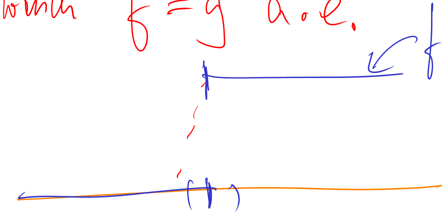
Theorem 6.29 (Lusin). *Let μ be a finite regular measure on a metric space X. Let $f : X \to \mathbb{R}$ be measurable. For any $\varepsilon > 0$ there exists a continuous function $g \colon X \to \mathbb{R}$ such that $\mu\{f \neq g\} < \varepsilon$.*

Cor (D.l.mv) : $\exists\, g : X \to \mathbb{R} \ni f = g$ a.e. & f is cts.

(False!)

Cor: f is cts a.e. (FALSE)

$\exists$ meas fns $\ni \nexists$ a cts fn g for which $f = g$ a.e.

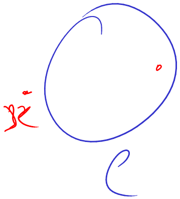


Lemma 6.30 (Tietze's extension theorem)**.** *If $C \subseteq X$ is ~~continuous~~ closed, and $f: C \to \mathbb{R}$ is continuous, then there exist $\bar{f}: X \to \mathbb{R}$ such that $\bar{f} = f$ on C.* AND $\bar{f}$ is cts

Rem: If X is a top space the the $\bar{f}$ is hard.

Pf:
$$\bar{f}(x) = \begin{cases} \inf\left\{c \in C \;\middle|\; f(c) + \frac{d(x,c)}{d(x,C)} - 1\right\} & x \notin C \\ f(x) & x \in C \end{cases}$$

ε–δ check $\bar{f}$ is cts.



Lemma 6.31. *Let $f\colon X \to \mathbb{R}$ be measurable. For every $\varepsilon > 0$, there exists $C \subseteq X$ closed such that $\mu(X - C) < \varepsilon$ and $f\colon C \to \mathbb{R}$ is continuous.*

Pf: Case I: $f : X \to [0,1]$ (f is bdd)

disj $\to$ $A_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \in \Sigma \Rightarrow \exists K_{n,k}$ cpt & $K_{n,k} \subset A_{n,k}$

& $\mu(A_{n,k} - K_{n,k}) < \frac{\varepsilon}{\cancel{2}^{n} 4^n}$

Let $C_n = \bigcup_k K_{n,k}$. Note $\mu(C_n^c) \leq 2^n \cdot \frac{\varepsilon}{\cancel{2^n} 4^n} = \frac{\varepsilon}{2^n}$

Let $C = \bigcap_1^\infty C_n$. Note C is closed & $\mu(C^c) \leq \sum \frac{\varepsilon}{2^n} = \varepsilon$

m

Claim: f is cts on C.

Pf: let $s_n = \sum_{k=0}^{2^n} \frac{k}{2^n} \underline{\mathbb{1}_{K_{n,k}}}$

Note s_n is cts on $\bigcup_k K_{n,k} = C_n \Rightarrow s_n$ cts on C

($K_{n,k}$ are disj as k varies).

Note $|s_{n+1} - s_n| \leq \frac{1}{2^{n+1}}$ on $C \Rightarrow (s_n) \longrightarrow f$ unif on C

$\Rightarrow f$ is cts on C

QED.

Case II: f not bdd

Set $g = \tan^{-1}(f)$

Proof of Lusin's theorem. Previous two lemmas. $\longrightarrow$ $\exists C$ closed & $f: C \longrightarrow \mathbb{R}$ is cts & $\mu(X-C) < \varepsilon$.

Use Tiezte to extend f. □

Proof of Lemma 6.31. □

7. Integration

$$\left(s = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}\right)$$

7.1. **Construction of the Lebesgue integral.** Recall, $s\colon X \to \mathbb{R}$ is *simple* if s is measurable and has finite range.

Definition 7.1. Let $s \geqslant 0$ be a simple function. Let $\{a_1, \ldots, a_n\} = s(X)$, and set $A_i = s^{-1}(a_i)$. Define $\int_X s\,d\mu = \sum_{i=1}^{n} a_i \cancel{A_i}\,\mu(A_i)$.

Remark 7.2. Always use the convention $0 \cdot \infty = 0$.

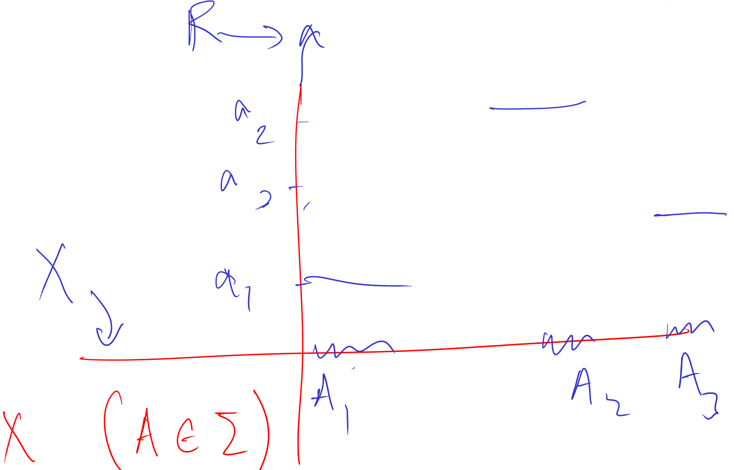
Remark 7.3. Other notation: $\int_X s\,d\mu = \int_X s(x)\,d\mu(x)$.

$$\int_X s\,d\mu = \sum_{i=1}^{n} a_i \underbrace{\mu(A_i)}$$

could be ∞

$$\int s = \int_X s\,d\mu$$

For MT only



Notation: $A \subseteq X$ $(A \in \Sigma)$

$$\int_A s\,d\mu = \int_X \mathbb{1}_A s\,d\mu = \sum_{i=1}^{n} a_i \mu(A \cap A_i)$$

Proposition 7.4. *If* $0 \leqslant s \leqslant t$ *are simple, then* $\int_X s\,d\mu \leqslant \int_X t\,d\mu$. (monotonicity of μ)

Proposition 7.5. *If* $s, t \geqslant 0$ *are simple, then* $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$.

(Same checking involved, straightforward)

Definition 7.6. Let $f\colon X \to [0, \infty]$ be measurable. Define $\int_X f\,d\mu = \sup\{\int_X s\,d\mu \mid 0 \leqslant s \leqslant f,\ s \text{ simple.}\}$. $(f \geq 0)$

Definition 7.7. Let $f\colon X \to [-\infty, \infty]$ be measurable. We say f is *integrable* if $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$. In this case we define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$. (lose cancellation!)

Definition 7.8. We let $L^1(X) = L^1(X, \Sigma, \mu)$ be the set of all integrable functions on X. (Note $f \in L^1 \iff |f| \in L^1$.)

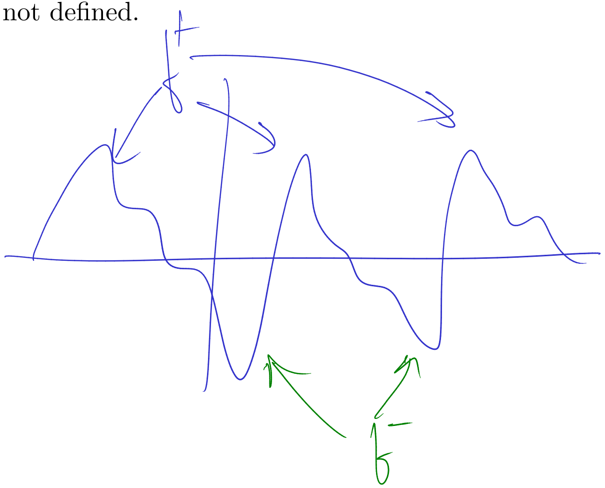
Definition 7.9. We say f is *integrable in the extended sense* if *either* $\int_X f^+\,d\mu < \infty$ or $\int_X f^-\,d\mu < \infty$. In this case we still define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Remark 7.10. If both $\int_X f^+\,d\mu = \infty$ and $\int_X f^-\,d\mu = \infty$, then $\int_X f\,d\mu$ is not defined.

Question 7.11. *Do we have linearity?*

$$f^+ = f \vee 0 = \max\{f, 0\}$$

$$f^- = -f \wedge 0 = -\min\{f, 0\}$$



Check linearity: $f, g \geq 0$.

$0 \leq s \leq f$, $0 \leq t \leq g$, s, t simple,

$\Rightarrow 0 \leq s+t \leq f+g$

$$\Rightarrow \int_X (f+g)\,d\mu \boxed{\geq} \sup\left\{ \int_X (s+t)\,d\mu \;\middle|\; \begin{matrix} 0 \leq s \leq f \\ 0 \leq t \leq g \end{matrix}, \ s,t \text{ simple} \right\}$$

$$= \int_X f\,d\mu + \int_X g\,d\mu.$$

Remk. If $\mu(X) < \infty$ & f, g bdd can show $\int_X (f+g)\,d\mu \leq \int_X f\,d\mu + \int_X g\,d\mu$

Proposition 7.12 (Consistency)**.** *If* $s = \sum_1^n a_i \mathbf{1}_{A_i} \geqslant 0$ *is simple, then* $\sum_{i=1}^{n} a_i\mu(A_i) = \sup\{\int_X t\,d\mu \mid 0 \leqslant t \leqslant s,\ \textit{simple}\}$.

① $t \le s \Rightarrow \int_X t\,d\mu \le \int_X s\,d\mu \Rightarrow \int_X s\,d\mu \geq \sup_{\substack{0\le t\le s \\ t \text{ simple}}} \int_X t\,d\mu$

② Choose $s = t$ & get equality.

Theorem 7.13 (Monotone convergence). *Say* $(f_n) \to f$ *almost everywhere,* $0 \leqslant f_n \leqslant f_{n+1}$, *then* $(\int_X f_n \, d\mu) \to \int_X f \, d\mu$.

Pf: ① $\lim_{n \to \infty} \int_X f_n \, dx$ exists (Yes. $\int f_n \leq \int f_{n+1}$)

(could be ∞)

② $\int_X f_n \, d\mu \leq \int_X f \, d\mu \Rightarrow \lim \int_X f_n \, d\mu \leq \int_X f \, d\mu$.

③ NTS $\liminf \int_X f_n \, d\mu \geq \int f \, d\mu$.

Pf: Let s simple, $0 \leq s \leq f$. enough to show $\lim \int f_n \geq \int s$

Let $E_n = \{ f_n \geq s(1-\varepsilon) \}$ Note $E_n \subseteq E_{n+1}$

Say $\bigcup E_n = X$. ← (OOPS)

Clearly $\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq (1-\varepsilon)\int_{E_n} s \, d\mu$.

$$= (1-\varepsilon)\sum_{i=1}^{m} a_i \, \mu(A_i \cap E_n) \qquad \left(s = \sum_{i=1}^{m} a_i \mathbb{1}_{A_i} \right)$$

$$\xrightarrow{n \to \infty} (1-\varepsilon)\sum_{i=1}^{m} a_i \, \mu(A_i) = (1-\varepsilon)\int_X s \, d\mu.$$

$\Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu \geq (1-\varepsilon)\int_X s \, d\mu \quad \forall \, s$ simple, $0 \leq s \leq f$. $\Rightarrow$ QED.

Theorem 7.14. *If* f, g *are integrable, then* $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$.

Pf: ① Say $f, g \geq 0$. Know $\exists\, s_n, t_n$ simple $\ni$

$$(s_n) \longrightarrow f, \quad (t_n) \longrightarrow g, \quad 0 \leq \underline{s_n \leq s_{n+1}} \quad 0 \leq t_n \leq t_{n+1}$$

M.C. $\Rightarrow$

$$\int_X (f+g)\,d\mu \overset{M.C}{=} \lim_{n\to\infty} \int_X (s_n + t_n)\,d\mu \underset{\substack{\uparrow \\ \text{linearity for} \\ \text{simple fns.}}}{=} \lim_{n\to\infty} \left(\int_X s_n\,d\mu + \int_X t_n\,d\mu \right)$$

$$\overset{M.C}{=} \int_X f\,d\mu + \int_X g\,d\mu$$

② Lemma: Say $f = g - h$ where $g \,\&\, h \geq 0$ $(f, g, h \in L^1)$

then $\int_X f\,d\mu = \int_X g\,d\mu - \int_X h\,d\mu$.

Pf: $f = f^+ - f^- = g - h \Rightarrow f^+ + h = f^- + g$ (all ≥ 0)

$\underline{\text{By}}$ ① $\Rightarrow \int_X f^+ + \int_X h = \int_X f^- + \int_X g$

$\Rightarrow \int_X f^+ - \int_X f^- = \int_X h - \int_X g$ QED.

③ NTS $f, g \in L^1$, NTS $\int (f+g) = \int f + \int g$.

Pf: $\underline{f+g} = (f^+ - f^-) + g^+ - g^- = \underbrace{(f^+ + g^+)}_{\geq 0} - \underbrace{(f^- + g^-)}_{\geq 0}$

By ② $\Rightarrow \int_X (f+g)\,d\mu = \int_X (f^+ + g^+)\,d\mu - \int_X (f^- + g^-)\,d\mu$

$$\overset{\text{by ①}}{=} \int_X f^+ + \int_X g^+ - \int_X f^- - \int_X g^-$$

$$= \left(\int_X f\,d\mu\right) + \left(\int_X g\,d\mu\right)$$

QED.

Last time: $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu.$

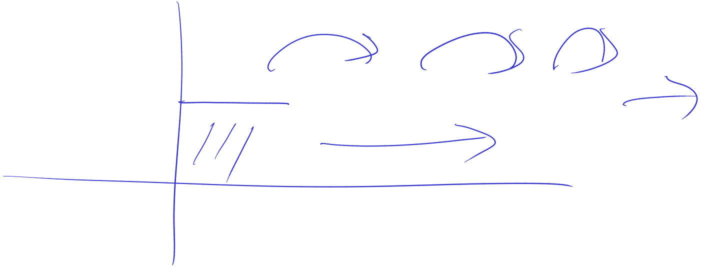
$f \geq 0$: $\int_X f\,d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_X s\,d\mu$ $\quad s = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$ & define $\int_X s\,d\mu = \sum a_k \mu(A_k)$

M.C.: $0 \leq f_n \leq f_{n+1}$, $(f_n) \longrightarrow f \Rightarrow \int_X f_n\,d\mu \longrightarrow \int_X f\,d\mu.$

7.2. **Dominated convergence.** When does $\underbrace{\lim \int_X f_n \, d\mu}_{} \neq \underbrace{\int_X f \, d\mu}_{}$? Two typical situations where it fails:

1. Mass escapes to infinity
2. Mass clusters at a point

$$f_n(x) = \mathbb{1}_{[n, n+1]}$$



$(f_n) \longrightarrow 0$ But $\int_{\mathbb{R}} f_n \, d\lambda = 1$

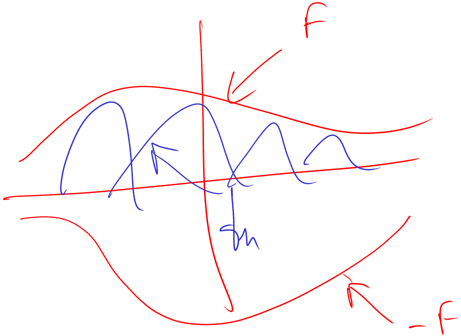
$$f_n(x) = n \, \mathbb{1}_{[0, \frac{1}{n}]}$$

① $(f_n) \longrightarrow 0$ (a.e.)

② $\int_{\mathbb{R}} f_n \, d\lambda = 1 \quad \forall n$

$\Rightarrow \underbrace{\lim \int_{\mathbb{R}} f_n \, d\lambda}_{1} \neq \int_{\mathbb{R}} f \, d\lambda = 0$

Theorem 7.15 (Dominated convergence)**.** *Say* (f_n) *is a sequence of measurable functions, such that* $(f_n) \to f$ *almost everywhere. Moreover, there exists* $F \in L^1(X)$ *such that* $|f_n| \leqslant F$ *almost everywhere. Then* $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.



Lemma 7.16 (Fatou). *Suppose* $f_n \overset{(a.e.)}{\geqslant} 0$, *and* $(f_n) \overset{(a.e.)}{\to} f$. *Then* $\liminf \int_X f_n \, d\mu \geqslant \int_X f \, d\mu$.

(+ve fns → mass can escape, but not be created)

Pf: Let $g_n = \inf_{k \geq n} f_k$, Note $0 \leq g_n \leq g_{n+1}$

$$\Rightarrow \text{By M.C.} \quad \lim_{n\to\infty} \int_X g_n \, d\mu = \int_X \left(\lim_{n\to\infty} g_n\right) d\mu = \int_X f \, d\mu.$$

$$\text{But} \quad g_n \leq f_n \Rightarrow \underbrace{\int_X g_n \, d\mu}_{\Big\downarrow n\to\infty} \leq \int_X f_n \, d\mu$$

$$\int_X f \, d\mu \qquad \longrightarrow \text{QED}.$$

Proof of Theorem $\boxed{7.15}$ D. C. $(f_n) \to f, \quad |f_n| \le F \in L^1(X)$

NTS $\lim \int_X f_n \, d\mu = \int_X f \, d\mu.$ $\quad \int_X F \, d\mu < \infty.$

Pf: ① Let $g_n = F + f_n \ge 0$

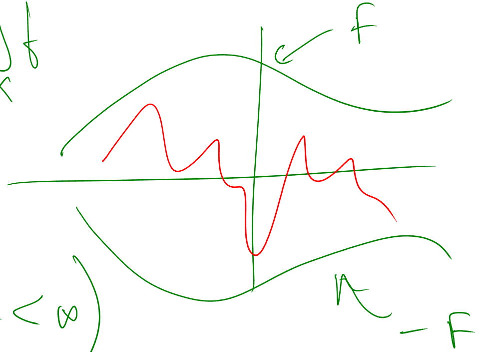
By Fatou: $\left.\begin{array}{l} \underline{\lim} \int_X g_n \, d\mu \ge \int_X (F + f) \, d\mu \\ \quad\quad \| \\ \underline{\lim} \int_X (F + f_n) \, d\mu \end{array}\right\} \Rightarrow \underline{\lim} \int_X f_n \, d\mu \ge \int_X f \, d\mu$

$(\because \int_X F \, d\mu < \infty).$

② $h_n = F - f_n \ge 0$

Fatou $\Rightarrow$ $\underline{\lim} \int_X (F - f_n) \geq \int_X F - \int_X f$

$\Rightarrow$ $\overline{\lim} \int_X f_n \, d\mu \leq \int_X f \, d\mu$



$\left(\because \int_X F \, d\mu < \infty \right)$

QED.

Theorem 7.17 (Beppo-Levi). *If* $f_n \geqslant 0$, *then* $\sum_1^\infty \int_X f_n \, d\mu = \int_X (\sum_1^\infty f_n) \, d\mu$.

Pf: $s_n = \sum_1^n f_k$ & use M. C.

Theorem 7.18. *If $f : \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, then the Riemann integral of f is the same as the Lebesgue integral.*

Proof. IOU ☐

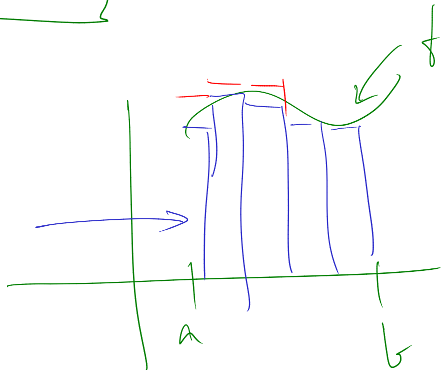
HW.

ε wiggling +

lower sum

=

$\int$ simple fn.



Question 7.19. *Let $f\colon [0,\infty) \to \mathbb{R}$ be measurable, and define the* Laplace transform *of f by $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Is F continuous? Is F differentiable?*

$f(t)$

$F(s)$

$\int_0^\infty e^{-st} f(t)\, d\lambda(t)$

Lebesgue.

Q1: Is F cts?

$s_n \to s$ (some seq). Want $F(s_n) \to F(s)$

Want $\int_0^\infty e^{-s_n t} f(t)\,dt \longrightarrow \int_0^\infty e^{-st} f(t)\,dt$

Clearly $\left(e^{-s_n t} f(t)\right) \longrightarrow \left(e^{-st} f(t)\right)$ $\forall t$

$\boxed{\text{Say } \int_0^\infty |f|\,dx < \infty.}$

Note: $\left|e^{-s_n t} f(t)\right| \leq |f(t)|$ $\forall n$

D.C. $\Rightarrow$ $\lim \int_0^\infty e^{-s_n t} f(t)\, dt = \int_0^\infty e^{-st} f(t)\, dt.$

$$\boxed{\therefore \text{① If } f \in L^1 \Rightarrow F \text{ is cts!}}$$

Q2: Is F diff?

s

Pick $s > 0$, $(s_n) \longrightarrow s.$

$$\frac{F(s_n) - F(s)}{s_n - s} = \int_0^\infty \underbrace{\left(\frac{e^{-s_n t} - e^{-st}}{s_n - s} \right) f(t)\, dt}_{g_n}.$$

Let $g_n(t) = \dfrac{e^{-s_n t} - e^{-st}}{s_n - s} f(t)$

Note $\big(g_n(t) \big) \longrightarrow -te^{-st} f(t)$

Q: Does $\displaystyle\int_0^\infty g_n(t)\,dt \longrightarrow -\int_0^\infty t e^{-st} f(t)\,dt$ ($\Rightarrow \underline{F}$ is diff at s)

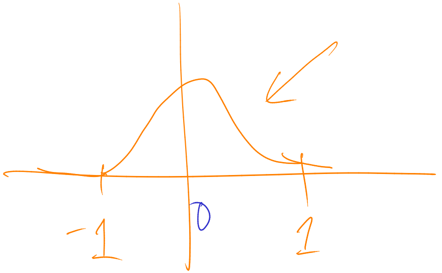
Note $$|g_n(t)| \leq \underbrace{\left| \frac{e^{-s_n t} - e^{-st}}{s_n - s} \right|} \, |f(t)|$$

$$\leq \underline{t\,|f(t)|} \quad (\text{M.V.T}).$$

If $t f(t) \in \mathcal{L}^1$

$\Rightarrow$ D.C. F is diff

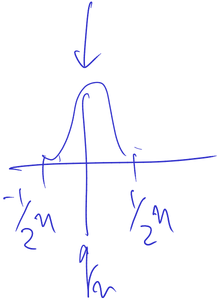
Question 7.20. *Let φ be a bump function, and (q_n) be an enumeration of the rationals. Define $f(x) = \sum_{n=1}^{\infty} \varphi(2^n(x-q_n))$. Is f finite almost everywhere?*



$\varphi \geq 0$

φ : cptly supp

$\int_{-\infty}^{\infty} \varphi = 1.$



$\int \varphi$

$$\int f\,dx \overset{BL}{=} \sum_{1}^{\infty} \int \underbrace{\varphi(2^n(x-q_n))}_{}$$

$$= \sum_{1}^{\infty} \frac{1}{2^n} < \infty$$

$\Rightarrow f < \infty$ a.e.!!

Last time: $\underbrace{\overset{\text{MC}}{\text{DCT}}}$

$\hookrightarrow$ $(f_n) \longrightarrow f$ a.e.

$|f_n| \leq F$ a.e. $\forall n$ (F ind of n)

& $\int_X F \, d\mu < \infty$

Then $\lim \int_X f_n \, d\mu = \int_X f \, d\mu$.

7.3. Push forward measures.

Definition 7.21. Say $f\colon X \to \mathbb{R}^d$ is integrable, then define $\int_X f\, d\mu = (\int_X f_1\, d\mu, \ldots, \int_X f_d\, d\mu$, where $f = (f_1, \ldots, f_d)$.

Theorem 7.22. *Let (X, Σ, μ) be a measure space, $f\colon X \to Y$ be arbitrary. Define $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$, and define $\nu(A) = \mu(f^{-1}(A))$. Then ν is a measure on (Y, τ) and $\int_Y g\, d\nu = \int_X g \circ f\, d\mu$.*

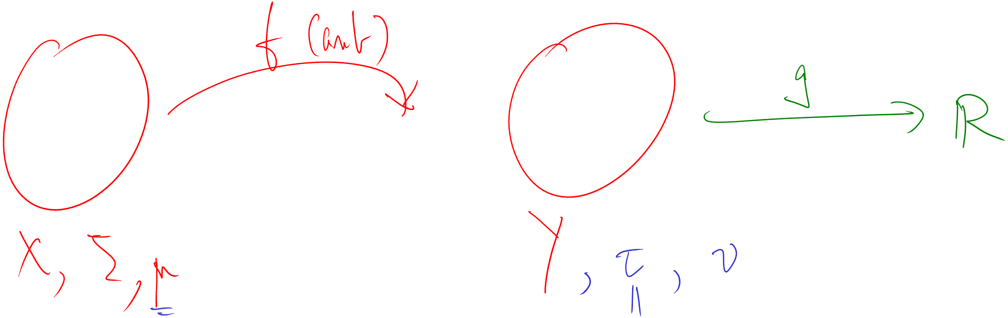
Remark 7.23. The measure ν is called the *push forward* of μ and denoted by $f^*(\mu)$, or $\mu_{f^{-1}}$. This is used often to define Laws of random variables. (We will use it to prove the change of variable formula.)

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right), \quad A_i \text{ disj}$$

$$= \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(A_i)\right)$$

$$= \sum_{i=1}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=1}^{\infty} \nu(A_i)$$



$$\{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$$

Prop ~~Q~~: Given $g : Y \to \mathbb{R}$. $\int_Y g\,d\nu = \int_X g \circ f\,d\mu$

Pf: Say $s : Y \to \mathbb{R}$ is simple.

$$s = \sum a_i \mathbb{1}_{A_i} \Rightarrow \int_Y s\,d\nu = \sum a_i \nu(A_i) = \sum a_i \mu(f^{-1}(A))$$

Also, $\int_X (s \circ f)\,d\mu = \int_X \sum a_i \mathbb{1}_{f^{-1}(A_i)}\,d\mu =$

$\Rightarrow \forall s$ simple, $\int_Y s\,d\nu = \int_X (s \circ f)\,d\mu$.

If $g : Y \to \mathbb{R}$ is ≥ 0

then find s_n simple $\ni$ $(s_n) \to g$, $0 \leq s_n \leq s_{n+1}$

$$\Rightarrow \underline{\int_Y \underline{g\,d\nu}} \overset{MC}{=} \lim \int_Y s_n\,d\nu = \lim \int_X (\underline{s_n \circ f})\,d\nu \overset{\underline{\underline{M.C.}}}{=} \int_X (g \circ \underline{f})\,d\mu.$$

QED.

Corollary 7.24. *If* $\alpha \in \mathbb{R}^d$, *then* $\int_{\mathbb{R}^d} f(x+\alpha)\, d\lambda(x) = \int_{\mathbb{R}^d} f(x)\, d\lambda(x)$.

$\hookrightarrow$ $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d \qquad g(x) = x + \underline{\alpha}$

Then $g^*(\lambda) = \lambda.$

By thm
$$\int_{\mathbb{R}^d} f\circ g \; d\lambda = \int_{\mathbb{R}} f \; d\underbrace{\left(g^*\lambda\right)}_{\lambda.} = \int_{\mathbb{R}^d} f(x)\, d\lambda(x)$$

$$\|$$

$$\int_{\mathbb{R}^d} f(x+\alpha)\, d\lambda(x)$$

QED.

8. Convergence

8.1. Modes of convergence.

Definition 8.1. We say $\boxed{(f_n) \to f}$ almost everywhere if for almost every $x \in X$, we have $(f_n(x)) \to f(x)$.

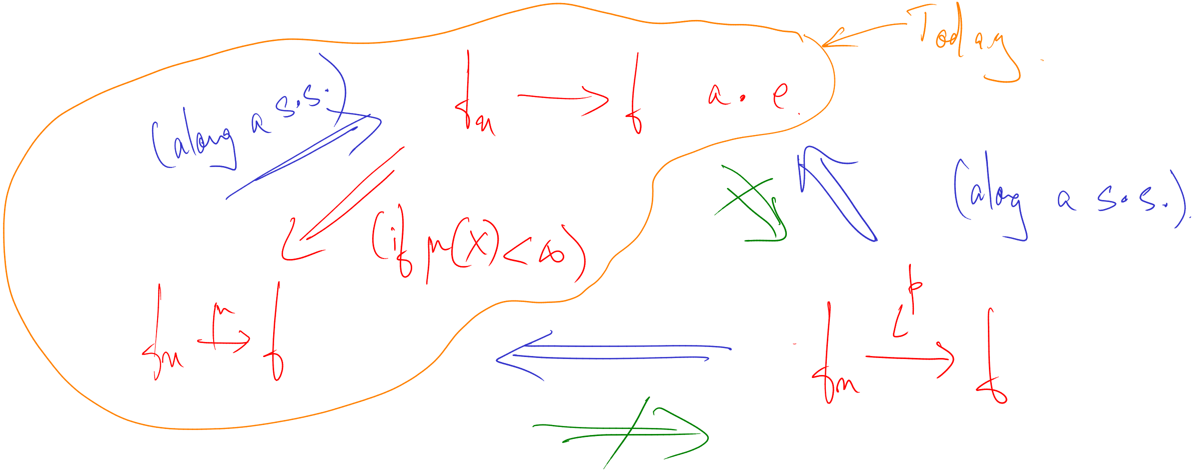
Definition 8.2. We say $(f_n) \to f$ in measure (notation $(f_n) \xrightarrow{\mu} f$) if for all $\varepsilon > 0$, we have $(\mu\{|f_n - f| > \varepsilon\}) \to 0$.

Definition 8.3. Let $\boxed{p \in [1, \infty)}$. We say $(f_n) \to f$ in L^p if $(\int_X |f_n - f|^p \, d\mu) \to 0$.

Question 8.4. *Why $p > 1$? How about $p = \infty$?*

$$\lim_{n \to \infty} \Big(\underbrace{\int |f_n - f|^p \, d\mu}_{\text{"dist between } f_n \text{ \& } f\text{"}} \Big)^{1/p} = 0$$

(1) $(f_n) \to f$ almost everywhere implies $(f_n) \to f$ in measure if $\mu(X) < \infty$.
(2) $(f_n) \to f$ in measure implies $(f_n) \to f$ almost everywhere along a subsequence.
(3) $(f_n) \to f$ in L^p implies $(f_n) \to f$ in measure (for $p < \infty$), and hence $(f_n) \to f$ along a subsequence.
(4) Convergence almost everywhere or in measure don't imply convergence in L^p.



Eg: $(f_n) \longrightarrow f$ a.e. but $(f_n) \not\longrightarrow f$ in meas

Chose $f_n = 1_{[n,\infty]}$ $\quad f = 0$

$$\left\{\begin{array}{l} (f_n) \longrightarrow 0 \text{ a.e.} \\ (f_n) \not\longrightarrow 0 \text{ in meas} \end{array}\right.$$

$$\left(\because \; \{|f_n - f| > \tfrac{1}{2}\} = \infty \;\; \forall n . \right)$$

Theorem 8.5. *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty}$, then $(f_n) \to f$ in measure.*

Lemma 8.6 (Egorov). *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty}$, for every $\varepsilon > 0$ there exists A_ε such that $(f_n) \to f$ uniformly on* ~~A_ε^c~~. A_ε & $\mu(A_\varepsilon^c) < \varepsilon$

Question 8.7. *Does this imply $(f_n) \to f$ uniformly almost everywhere?* (No : $f_n(x) = x^n$ $[0,1]$.)

Pf of Egorov: $\forall\, k \in \mathbb{N}$, $\bigcup_{m=1}^{\infty} \bigcap_{n \geq m} \{ |f_n - f| \leq \frac{1}{k} \} = X -$ null set

& this is an inc union. $\Rightarrow \lim_{m \to \infty} \mu\left(\bigcap_{n \geq m} \{ |f_n - f| < \frac{1}{k} \} \right) = \mu(X)$

$\Rightarrow \forall k, \exists\, n_k \ni \mu\left(\underbrace{\bigcap_{n \geq n_k} \{ |f_n - f| < \frac{1}{k} \}}_{A_k} \right) \geq \mu(X) - \frac{\varepsilon}{2^k}$

Let $A = \bigcap_{k=1}^{\infty} A_k$. ① Note $\mu(A^c) \leq \sum_{1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$

② Note : $f_n \longrightarrow f$ unif on A.

$(\because \forall n \geq n_k , \ |f_n - f| \leq \frac{1}{k} \ \forall x \in A_k \supseteq A)$

QED.

Theorem 8.5. *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty,}$ then $(f_n) \to f$ in measure.*

Lemma 8.6 (Egorov). *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty,}$ for every $\varepsilon > 0$ there exists A_ε such that $\mu(A_\varepsilon^c) < \varepsilon$ and $(f_n) \to f$ uniformly on A_ε.*

Question 8.7. *Does this imply $(f_n) \to f$ uniformly almost everywhere?*

→ Egorov → last time.

Thm 8.5: Pick $\varepsilon > 0$. NTS $\mu(|f_n - f| > \varepsilon) \longrightarrow 0$

Egorov $\Rightarrow \forall \delta > 0, \exists A_\delta$ s.t. $\mu(A_\delta^c) < \delta$

& $f_n \to f$ unif on A_δ.

$\Rightarrow \{|f_n - f| > \varepsilon\} \subseteq A_\delta^c$ $\forall$ large n.

$(f_n) \xrightarrow{\mu} f$

if $\forall \varepsilon > 0$

$\mu(|f_n - f| > \varepsilon) \xrightarrow{n \to \infty} 0$

Proof of Theorem 8.5

$$\Rightarrow \mu(\{|f_n - f| > \varepsilon\}) \leq \mu(A_\delta^c) = \delta$$

QED.

$$\begin{tikzcd}
 & f_n \to f \text{ a.e.} \arrow[dl, Rightarrow, shift right, "(\mu(X)<\infty)"'] \arrow[dr, Rightarrow, "/" marking] & \\
f_n \to f \text{ meas} \arrow[ur, Rightarrow, shift right, "\text{along s.s.}"] \arrow[rr, Rightarrow, shift right, "/" marking] & & f_n \to f \ L^p \arrow[ul, Rightarrow, "\text{along s.s.}"'] \arrow[ll, Rightarrow, shift right, "(p<\infty)"']
\end{tikzcd}$$

Proposition 8.8. *If* $\underline{(f_n) \to f}$ *in measure then* (f_n) *need not converge to* f *almost everywhere.*

Eg: Pf by Picture $f_1 = \mathbb{1}_{[0, 1/2]}$ $f_2 = \mathbb{1}_{[1/2, 1]}$

$f_3 = \mathbb{1}_{[0, 1/4]}$, $f_4 = \mathbb{1}_{[1/4, 1/2]}$, $f_5 = \mathbb{1}_{[1/2, 3/4]}$ $f_6 = \mathbb{1}_{[3/4, 1]}$

$f_7 = \mathbb{1}_{[0, 1/8]}$, . . . $f_8 = \mathbb{1}_{[1/8, \frac{2}{8}]}$ etc.

Q: $(f_n) \longrightarrow 0$ in measure $\left(\mu(|f_n - 0| > \varepsilon) \longrightarrow 0 \ \forall \varepsilon \right)$

Q: $(f_n) \longrightarrow 0$ ptwise? NO. $\forall x$ $f_n(x) = 0$ i.o. & $f_n(x) = 1$ i.o.

Proposition 8.9. *If $(f_n) \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere.*

Pf: $\forall k \in \mathbb{N}$, $\mu(|f_n - f| > \frac{1}{k}) \xrightarrow{n \to \infty} 0$

$\forall k$, $\exists n_k \ni n_k > n_{k-1}$ & $\mu(|f_{n_k} - f| > \frac{1}{k}) \leq \frac{1}{2^k}$

Let $A_k = \{|f_{n_k} - f| > \frac{1}{k}\}$.

Let $B = \{x \mid x$ only $\in$ finitely many $A_k\}$

① $\forall x \in B$, $|f_{n_k}(x) - f(x)| \leq \frac{1}{k}$ $\forall$ large k $\Rightarrow (f_{n_k}(x)) \longrightarrow f(x)$

② NTS $\mu(B^c) = 0$

$$B^c = \{x \mid x \in \infty^{ly} \text{ many } A_k\}$$

$$= \{x \mid \forall m \ \exists n \geq m \ \ni \ x \in A_n\}$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n$$

$$\Rightarrow \forall m, \ \mu(B^c) \leq \mu\Big(\bigcup_{n \geq m} A_n\Big) \leq \sum_{n \geq m} \mu(A_n) = \sum_{n \geq m} \frac{1}{2^n} = \frac{1}{2^m}$$

$$\mu(B^c) \leq \frac{1}{2^m} \ \forall m \quad \Rightarrow \mu(B^c) = 0$$

QED.

8.2. L^p spaces.

Definition 8.10. A *Banach space* is a normed vector space that is complete under the metric induced by the norm.

Example 8.11. $\mathbb{C}$, $\mathbb{R}^d$, $C(X)$, etc. are all Banach spaces.

$C(X) = \{f : X \to \mathbb{R} \text{ cts}\}$, X cpt.

Definition 8.12. For $p \in (0, \infty)$, define $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

$\|f\| = \sup_{x \in X} |f(x)|$

Definition 8.13. For $p = \infty$, define $\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\{C \geqslant 0 \mid |f| \leqslant C \text{ almost } \cancel{\text{surely}} \text{ everywhere}\}$

Definition 8.14. Let (X, Σ, μ) be a measure space, and assume Σ is μ-complete. Define $\mathcal{L}^p(X) = \{f : X \to \mathbb{R} \mid \|f\|_p < \infty\}$.

Question 8.15. *Is $\mathcal{L}^p(X)$ a Banach space?*

$X \to$ normed V.S.

$\|\cdot\| : X \to [0, \infty)$ +

⓪ $\|x\| = 0 \iff x = 0$

① $\|\lambda x\| = |\lambda| \, \|x\|$

& ② $\|x + y\| \leq \|x\| + \|y\|$

dist: $d(x, y) = \|x - y\|$

(X, d) metric

$p=\infty$: $\|f\|_\infty =$ " $\sup\limits_{x\in X} |f(x)|$ " $= \sup\{a \mid \mu\{|f| \geq a\} > 0\}$.

ess sup

$\longrightarrow = \inf\{c \mid |f| \leq c \text{ a.e.}\}$

$(-\|f\|_\infty \leq f \leq \|\underline{f}\|_\infty$ a.e.).

$\mathcal{L}^p(X) = \{f \mid \|f\|_p < \infty\}$. ← Not a Banach space.

$\underline{f = 0 \text{ a.e.}}$, but $f \neq 0$, we still have $\|f\|_p = 0$

Definition 8.16. Define an equivalence relation on $\mathcal{L}^p$ by $f \sim g$ if $f = g$ almost everywhere.

$L^p(X)$

Definition 8.17. Define $L^p(X) = \mathcal{L}^p(X)/\sim$.

$$L^p(X) = \mathcal{L}^p(X)/\sim$$

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

Theorem 8.19. *For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.*

$\leftarrow$ IOU Proof.

Hard details to check ① $\triangle$ ineq ② Completeness

$$L^p(X) = \{ f \mid \|f\|_p < \infty \}$$

$f \in L^p$. implicitly mean class of all g s.t. $g = f$ a.e.

↳ $f(x_0)$, $x_0 \in X$. $f(x_0) \longrightarrow$ not defined on L^p.

$f \in L^p \longrightarrow \int_A f \, d\mu \leftarrow$ OK, $\mu\{|f| > \lambda\} \leftarrow$ OK

Theorem 8.20 (Hölder's inequality). *Say $\underline{p, q} \in [1, \infty]$ with $\boxed{1/p + 1/q = 1}$. If $\underline{f \in L^p}$ and $\underline{g \in L^q}$, then $\underline{fg} \in L^1$ and* $|\underline{\int_X fg\, d\mu}| \leqslant \|f\|_p \|g\|_q$.

Hölder conjugates.

Remark 8.21. The relation between p and q can be motivated by dimension counting, or scaling.

Motivation → "Dimension counting".

$f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ (dimensionless)

$L \to$ length (dimension).

Q: dim of $\int_{\mathbb{R}^d} f g \, d\lambda = L^d$

$\dim \|f\|_p = \dim \left(\int_{\mathbb{R}^d} |f|^p dx \right)^{1/p}$

$= L^{d/p}$

$\dim \|f\|_q = L^{d/q}$

$\dim \left(\|f\|_p \, \|g\|_q \right) = L^{\boxed{\frac{d}{p} + \frac{d}{q}}}$

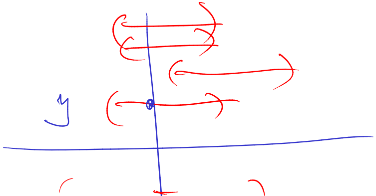
$= d$

Example of a regular meas in a non σ-finite space

$X = \mathbb{R}$ (λ meas)

$Y = \mathbb{R}$ (counting meas, discrete top)

$X \times Y \longrightarrow$ open sets



If $\lambda(U_y) > 0$ for uncountably many y, then $\sum(\ ) = \infty$

μ : $\underline{U} \subseteq X \times Y$ open

$$U = \bigcup_{y \in \mathbb{R}} U_y \times \{y\}$$

($U_y \subseteq X$ open) define $\mu(U) = \sum_{y \in \mathbb{R}} \lambda(U_y)$

$A \subseteq X \times Y$ Borel, $\underline{\mu(A)} = \inf\{\mu(U) \mid U \supseteq A \text{ open}\}$.

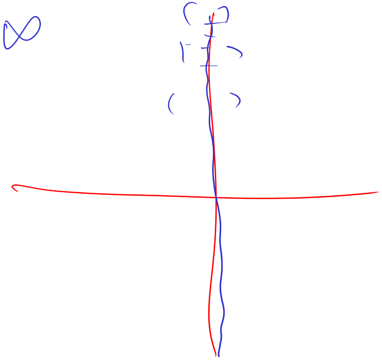
You check μ gives a regular measure on $X \times Y$.

Q: What is the measure of $\{0\} \times Y$. $= \infty$

Q: $K \subseteq \{0\} \times Y$ is cpt

What is $\mu(K) = 0$

(Gives necessary C-ege on HW 5/6).



Definition 8.16. Define an equivalence relation on $\underline{\mathcal{L}^p}$ by $\boxed{f \sim g}$ if $f = g$ almost everywhere.

Definition 8.17. Define $\mathcal{L}^p(X) = \mathcal{L}^p(X)/\sim$.

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

Theorem 8.19. *For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.*

$$\|f\|_p = \left(\int_X |f|^p\right)^{1/p}.$$

Theorem 8.20 (Hölder's inequality)**.** *Say* $p, q \in [1, \infty]$ *with* $1/p + 1/q = 1$*. If* $f \in L^p$ *and* $g \in L^q$*, then* $fg \in L^1$ *and* $|\int_X fg\, d\mu| \leqslant \|f\|_p \|g\|_q$.

Remark 8.21. The relation between p and q can be motivated by dimension counting, or scaling.

Let $\varepsilon > 0$. $X = \mathbb{R}^d$. $f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$, $g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$

① $$\int_{\mathbb{R}^d} f_\varepsilon\, g_\varepsilon = \varepsilon^d \int_{\mathbb{R}^d} f\left(\frac{x}{\varepsilon}\right) g\left(\frac{x}{\varepsilon}\right) \frac{dx}{\varepsilon^d} = \varepsilon^d \int_{\mathbb{R}^d} f\, g\, d\lambda.$$

② $$\|f_\varepsilon\|_p = \left(\varepsilon^d \int_{\mathbb{R}^d} \left|f\left(\frac{x}{\varepsilon}\right)\right|^p \frac{dx}{\varepsilon^d}\right)^{1/p} = \varepsilon^{d/p} \|f\|_p$$

③ $$\|g\|_q = \quad \cdots \quad = \varepsilon^{d/q} \|g\|_q.$$

If Holder is true $\Rightarrow$

$$\begin{array}{ccc} \left|\int_{\mathbb{R}^d} f_\varepsilon g_\varepsilon \, dx\right| & \leq & \|f_\varepsilon\|_p \, \|g_\varepsilon\|_q \\ \parallel & & \parallel \\ \varepsilon^d \left|\int_{\mathbb{R}^d} f g\right| & & \varepsilon^{\frac{d}{p} + d/q} \|f\|_p \, \|g\|_q \end{array}$$

Since this is true $\forall \varepsilon$, must have $d = \frac{d}{p} + \frac{d}{q} \iff \boxed{\frac{1}{p} + \frac{1}{q} = 1}$

Brute force proof of Theorem 8.20 :

Stupid.

⇒ ① Induction $\sum_{1}^{N} x_i y_i \le \left(\sum_{1}^{N} x_i^p\right)^{1/p} \left(\sum_{1}^{N} y_i^q\right)^{1/q}$.

$(x_i, y_i \ge 0)$.

② Say $c_i \ge 0$, $\sum x_i y_i c_i = \sum \underbrace{x_i c_i^{1/p}}_{} \; y_i c_i^{1/q}$

$$\le \left(\sum x_i^p c_i\right)^{1/p} \left(\sum y_i^q c_i\right)^{1/q}$$

③ ⇒ Holder is true for simple fns.

④ Approximate ⇒ QED.

Proof of Theorem 8.20 using Young's inequality.

Theorem 8.22 (Young's inequality). *If* $x, y \geqslant 0$, $\boxed{1/p + 1/q = 1}$ *then* $xy \leqslant x^p/p + y^q/q$.

Pf 1: Calculus min ( )

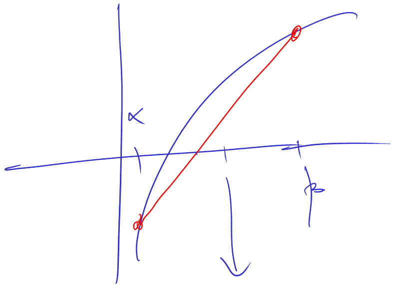
Pf 2: $\ln x$ is concave & increasing.

$\Rightarrow c \in (0,1), \quad \alpha, \beta > 0$

$$c \ln \alpha + (1-c) \ln \beta \leq \ln(c\alpha + (1-c)\beta)$$

$$\| \quad \ln(\alpha^c \beta^{1-c}).$$

$$\Rightarrow \quad \alpha^c \beta^{1-c} \leq c\alpha + (1-c)\beta.$$



$c\alpha + (1-c)\beta.$

$c = \frac{1}{p}. \qquad 1-c = \frac{1}{q}.$

$\alpha = x^p \qquad \beta = y^q$

QED.

Pf of Holder: NTS $\left| \int_X fg \right| \le \|f\|_p \|g\|_q$.

Let $\tilde{f} = \dfrac{f}{\|f\|_p}$ & $\tilde{g} = \dfrac{g}{\|g\|_q}$.

$(\|\tilde{f}\|_p = 1 \qquad \|\tilde{g}\|_q = 1)$

Holder $\Longleftrightarrow$ showing $\left| \int_X \tilde{f}\tilde{g} \right| \le 1$

Pf of:

$$\left| \int_X \tilde{f}\,\tilde{g}\,d\mu \right| \le \int_X |\tilde{f}|\,|\tilde{g}|\,d\mu \overset{\text{Young}}{\le} \int_X \left(\frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} \right) d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

QED.

(You check $p = 1, q = \infty$)

Lemma 8.23 (Duality). *If* $p \in [1, \infty)$, $1/p + 1/q = 1$, *then* $\|f\|_p = \sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg\, d\mu = \sup_{\|g\|_q = 1} \int_X fg\, d\mu \cancel{= 1}$

Remark 8.24. For $p = \infty$ this is still true if X is σ-finite.

Pf of Duality: ① $\frac{1}{\|g\|_q} \int_X fg\, d\mu \leq \|f\|_p$ (Holder).

$$\Rightarrow \sup_{g \in L^q - \{0\}} \frac{1}{\|g\|_q} \int_X fg\, d\mu \leq \|f\|_p .$$

② NTS equality. Chose $g = |f|^{p-1} \operatorname{sign}(f) \Rightarrow fg = |f|^p .$

$$\|g\|_q^q = \int_X |g|^q\, d\mu = \int_X |f|^{pq-q} = \int_X |f|^p$$

u

$$\boxed{\Leftrightarrow \|g\|_q^q = \|f\|_p^p}$$

$$\boxed{\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow pq = p+q}$$

$$\Rightarrow \frac{1}{\|g\|_q}\int_X fg \, d\mu = \frac{1}{\|g\|_q}\int_X |f|^p \, d\mu = \frac{\|f\|^p}{\|g\|_q} = \|f\|_p^{p\left(1-\frac{1}{q}\right)} = \|f\|_p$$

QED.

Theorem 8.25 (Minkowski's inequality)**.** *If* $\underline{\underline{f, g}} \in L^p$, *then* $\underline{f + g} \in L^p$ *and* $\underbrace{\|f + g\|_p \leqslant \underline{\|f\|_p + \|g\|_p}}$.

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$$

$$L^p = \{ f \mid \|f\|_p < \infty \}$$

IOU: L^p is a Banach space.

① $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

② Completeness.

(last time).

Holder: $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$, $\left|\int_X fg\right| \leq \|f\|_p \|g\|_q$.

Lemma 8.23 $\boxed{\text{(Duality)}}$. *If* $p \in [1, \infty)$, $1/p + 1/q = 1$, *then* $\|f\|_p = \sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg \, d\mu = \sup_{\|g\|_q = 1} \int_X fg \, d\mu = 1$

Remark 8.24. For $p = \infty$ this is still true if X is σ-finite.

Rmk: If $\sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg \, d\mu < \infty$ then $f \in L^p$ &

$Y \rightarrow$ Banach space. $Y^* =$ dual of Y $= \{ y^* : Y \rightarrow \mathbb{R} \mid y^*$ is linear & cts$\}$.

Norm on Y^*: define $\|y^*\| = \sup_{\|y\|=1} y^*(y) = \sup_{y \in Y - 0} \frac{y^*(y)}{\|y\|}$

$\boxed{\text{IOU: } (L^p)^* = L^q \quad (1 \leq p < \infty)}$

Theorem 8.25 (Minkowski's inequality). *If $f, g \in L^p$, then $f+g \in L^p$ and $\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$.*

$(p<\infty)$

Pf: $\|f+g\|_p = \sup_{\|h\|_q=1} \int_X (f+g)\, h \, d\mu \le \sup_{\|h\|_q=1} \int_X f h \, d\mu + \sup_{\|h\|_q=1} \int_X g h \, d\mu$

$= \|f\|_p + \|g\|_p$ QED.

$p=\infty \rightarrow$ easy.

(Note $f, g \in L^p \Rightarrow f+g \in L^p$: Pf $\left(\frac{1}{2}f + \frac{1}{2}g\right)^p \le \frac{1}{2}|f|^p + \frac{1}{2}|g|^p$

$\Rightarrow |f+g|^p \le 2^{p-1}\left(|f|^p + |g|^p\right) \leftarrow$ integrable

$\rightarrow f+g \in L^p$ QED.)

Theorem 8.26 (Jensen's inequality). *If* $\boxed{\mu(X) = 1}$, $f \in L^1(X)$, $a < f < b$ *almost everywhere, and* $\varphi : (a,b) \to \mathbb{R}$ *is* $\boxed{\text{convex,}}$ *then*

$$\varphi\Big(\int_X f\,d\mu\Big) \leqslant \int_X \varphi \circ f\,d\mu\,.$$

$(a, b \in [-\infty, \infty])$

Bad Proof:

φ convex, $\left.\begin{array}{l} c_i \in [0,1] \text{ \& } \sum_1^N c_i = 1 \\ x_i \in (a,b) \end{array}\right\} \Rightarrow \varphi\left(\sum c_i x_i\right) \leq \sum c_i \varphi(x_i)$

(def of convexity)

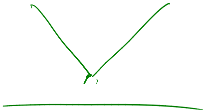
$\Rightarrow$ Jensen is true for simple fns!

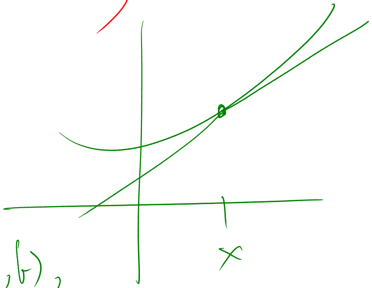
(i.e. $s = \sum a_i \mathbb{1}_{A_i} \Rightarrow \int_X s = \underbrace{\sum a_i \mu(A_i)}_{\text{convex linear comb of } a_i\text{'s.}}$)

Note $\sum \mu(A_i) = 1$

& use red ineq with $c_i = \mu(A_i)$. $\Rightarrow$ done

(3) And fns : approximate. (requires same care)





Better Proof: φ convex $\Rightarrow$

$$\varphi(x) + (y-x)\,\varphi'(x) \leq \varphi(y) \quad \forall y \in (a,b),\ \forall x \in (a,b)$$

$$\Rightarrow \int_X \left[\varphi(x) + (f(y)-x)\,\varphi'(x)\right] d\mu(y) \leq \int_X \varphi(f(y))\, d\mu(y) \quad \forall y \in X$$

$$\Rightarrow \varphi(x) + \varphi'(x) \int_X (f(y)-x)\, d\mu(y) \leq \int_X \varphi \circ f(y)\, d\mu(y)$$

Choose $x = \int_X f \, d\mu \Rightarrow \varphi\left(\int_X f \, d\mu\right) + \varphi'(x)\left(0\right) \leq \int_X \varphi \circ f \, d\mu(y)$

QED.

Completeness

Proof of Theorem $\boxed{8.19}$: Only remains to show L^p is complete.

Lemma 8.27. *Suppose* $p < \infty$, $f_n \in L^p$ *and* $\sum \|f_n\|_p < \infty$. *Let* $f = \sum f_n$. *Then* $f \in L^p$, *and* $\sum f_n \to f$ *in* L^p *and* $\sum f_n \to f$ *almost everywhere.*

Pf: ① Let $F = \sum_{1}^{\infty} |f_n|$.

Let $S_N = \sum_{1}^{N} |f_n| \in L^p$. Know $\|S_N\|_p \leq \sum_{1}^{N} \|f_n\|_p \longrightarrow \sum_{1}^{\infty} \|f_n\|_p < \infty$

$$\Rightarrow \|S_N\|_p^p \longrightarrow \left(\sum_{1}^{\infty} \|f_n\|_p\right)^p < \infty.$$

$$\int_X |S_N|^p \, d\mu$$

$$\Rightarrow F \in L^p \quad \left(\because \int_X F^p \, d\mu \overset{M.C.}{=} \lim_{N \to \infty} \int_X S_N^p \, d\mu < \infty\right)$$

② $\Rightarrow f \in L^p$, $\Rightarrow \sum_1^\infty |f_n| < \infty$ a.e.

$\Rightarrow \sum_1^\infty f_n$ is cgt a.e. ($\Rightarrow$ ~~claim~~ 2nd assertion).

③ Let $f = \sum_1^\infty f_n$. NTS $\left(\sum_1^N f_n\right) \longrightarrow f$ in L^p.

Note $f - \sum_1^N f_n = \sum_{N+1}^\infty f_n$

$$\Rightarrow \left\| f - \sum_1^N f_n \right\|_p \le \sum_{N+1}^\infty \| f_n \|_p \xrightarrow{N \to \infty} 0$$

QED.

Proof of Theorem 8.19: (L^p is complete).

Pf: Suppose (f_n) is a Cauchy Seq in L^p

$\exists\, n_k$ s.t. $\|f_{n_{k+1}} - f_{n_k}\| \le 2^{-k}$

By lemma, $\sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$ is cgt in L^p

$\Leftrightarrow (f_{n_k})$ is cgt in L^p.

Since (f_n) is Cauchy & (f_{n_k}) is cgt $\Rightarrow (f_n)$ is cgt QED.

HW ~~6~~ 5: Q 6

① $U = \bigcup_{1}^{\infty} (a_k, b_k)$

② $\lim_{n \to \infty} (a_k, b_k) \cap A_n = \frac{b_k - a_k}{2}$ $\Big\}$?

$$\Longrightarrow \lim_{n \to \infty} \sum_{1}^{\infty} (a_k, b_k) \cap A_n$$

$$\|$$

$$\underline{\left(\sum_{k=1}^{\infty} \cancel{a} \frac{b_k - a_k}{2} \right.}$$

Proposition 8.28. *If $p \in [1, \infty)$, $(f_n) \to f$ in L^p, then $(f_n) \to f$ in measure.*

Lemma 8.29 (Chebychev's inequality)**.** *For any $\lambda > 0$, we have $\mu(\{|f| > \lambda\}) \leqslant \dfrac{1}{\lambda}\|f\|_1$*

Pf: $$\int_X \lambda \mathbb{1}_{\{|f|>\lambda\}} \, d\mu \le \int_X |f| \mathbb{1}_{\{|f|>\lambda\}} \, d\mu \le \|f\|_{L^1}$$

$$\|$$

$$\lambda \mu\{|f|>\lambda\} \quad \Rightarrow \mu(|f|>\lambda) \le \frac{1}{\lambda}\|f\|_1 \quad \text{QED.}$$

Cor: $\forall p \ge 1$, $\mu\{|f|>\lambda\} \le \frac{1}{\lambda^p}\|f\|_p^p$ (Pf: $\{|f|>\lambda\} = \{|f|^p > \lambda^p\}$ & chebychev)

Proof of Proposition 8.28

If $(f_n) \longrightarrow f$ in L^p $\Rightarrow$ $\|f_n - f\|_p \xrightarrow{n\to\infty} 0$

$\Rightarrow \forall \varepsilon > 0$ $\mu\{|f_n - f| > \varepsilon\} \leq \frac{1}{\varepsilon^p}\|f_n - f\|_p^p \xrightarrow{n\to\infty} 0$

QED.

Converse? $(f_n) \longrightarrow f$ in meas does $(f_n) \longrightarrow f$ in L^p

No: $f_n = \frac{1}{n}\mathbb{1}_{[0,n^2]} \longrightarrow 0$ in meas

$\not\longrightarrow 0$ in L^1

8.3. Uniform integrability.

Question 8.30. *When does convergence in measure imply L^1 convergence?*

Definition 8.31. We say $\{f_\alpha | \alpha \in \mathcal{A}\}$ is uniformly integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$ we have $\int_E |f| \, d\mu < \varepsilon$.

Proposition 8.32. *If $|f_\alpha| \leqslant F$ for all $\alpha \in \mathcal{A}$, and $F \in L^1$, then $\{f_\alpha \mid \alpha \in A\}$ is uniformly integrable.* (Dominated $\Rightarrow$ U.I.)

Remk: U.I. $\not\Rightarrow$ Integrable! (Eg. $f(x) = 1$ $\forall x \in \mathbb{R}$, $\{f\}$ is U.I.)

Pf: Note: $\displaystyle \lim_{\lambda \to \infty} \int_{\{F > \lambda\}} F \, d\mu = \lim_{\lambda \to \infty} \int_X \mathbb{1}_{\{F > \lambda\}} F \, d\mu \xrightarrow[\text{(or M.C)}]{\text{D.C.}} 0$

Let $\varepsilon > 0$. Choose $\delta =$ ______

Assume $\mu(E) < \delta$:

$$\int_E |f_n|\, d\mu \le \int_E F = \int_{\underbrace{E\cap\{F>\lambda\}}} F + \int_{E\cap\{F\le\lambda\}} F$$

$$\le \underbrace{\int_{\{F>\lambda\}} F}_{\lambda\to\infty} \xrightarrow{\quad} 0 \qquad + \lambda\delta$$

Choose λ large + $\int_{\{f>\lambda\}} F < \frac{\varepsilon}{2}$. Choose $\delta \le \frac{\varepsilon}{2\lambda}$ $\Rightarrow$ Done

QED.

Theorem 8.33 (Vitali). *Let $\boxed{(f_n) \in L^1(X)}$. The sequence (f_n) is convergent in L^1 if and only if*

1. *(1) (f_n) converges in measure,*
2. *(2) (f_n) is uniformly integrable,*
3. *(3) (tightness) $\forall\varepsilon$ there exists $F \in \Sigma$ with $\boxed{\mu(F) < \infty}$ such that $\int_{F^c} |f_n| \, d\mu < \varepsilon$ for all n.*

Proof:

(Note: $\mu(X) < \infty \Rightarrow$ Tightness is automatic).

Pf $\Rightarrow$: Assume $(f_n) \longrightarrow f$ in L^1

① $(f_n) \to f$ in meas (Done $\to$ Chebychev).

② $\{f_n\}$ is U. I.

Pf: Pick $\varepsilon > 0$. Want $\delta > 0 \ni \mu(E) < \delta \Rightarrow \forall n \int_E |f_n| < \varepsilon$

$$\int_E |f_n| = \underbrace{\int_E |f_n - f|}_{\le \underbrace{\int_X |f_n - f|}_{\le \varepsilon.}} + \underbrace{\int_E |f|}_{}$$

make this $< \varepsilon/2$.

$\{f\}$ is U.I if f is L'

① find n_0 ∋ $\|f_n - f\| < \varepsilon \quad \forall n > n_0$

② $\{f_1, f_2, \cdots f_{n_0}, f\}$ is U.I. (∵ dominated by $\max_{i \le n_0} |f_i| \vee |f|$)

③ $\Rightarrow \exists \delta_1 > 0$ ∋ if $i \le n_0$, & $\mu(E) < \delta \Rightarrow \int_E |f_i| < \varepsilon/2$

& $\int_E |f| < \varepsilon/2$.

$④ \Rightarrow \int_E |f_n| \rightsquigarrow ①\ n \le n_0 \longrightarrow$ done by $\uparrow$

$② \ n \ge n_0 : \int_E |f_n| \le \int_E |f_n - f| + \int_E |f|$

$$\le \underbrace{\int_X |f_n - f|}_{\le \varepsilon/2} + \underbrace{\int_E |f|}_{\le \varepsilon/2}$$

$(\because n \ge n_0)$

QED.

③ Check tightness: NTS $\forall \varepsilon > 0$, $\exists E \ni \mu(E) < \infty$ & $\int_{E^c} |f_n| < \varepsilon$ $\forall n$.

Scratch: Q1: Show $\forall \varepsilon > 0$, $\exists E \ni \mu(E) < \infty$ & $\int_{E^c} |f| < \varepsilon$

$$\left.\begin{array}{l} \mu\{|f| > \delta\} \leq \frac{1}{\delta}\|f\|_1 < \infty \quad \forall \delta > 0 \\ \lim\limits_{\delta \to 0} \int_{|f| \leq \delta} |f| \, d\mu \overset{D.C.}{=} 0 \end{array}\right\} \Rightarrow$$

$\forall \varepsilon > 0$, $\exists \delta \ni$ For $E = \{f > \delta\}$, we have $\mu(E) < \infty$ & $\int_{E^c} |f| \, d\mu < \varepsilon$

Q2: If $f_1, f_2 \cdots f_{n_0}$ are finitely many fns,

$\forall \varepsilon > 0$, $\exists E \ni \mu(E) < \infty$ & $\int_{E^c} |f_i| < \varepsilon \ \forall i \le n_0$

Pf of tightness: $\forall \varepsilon > 0$, $\exists n_0 \ni \int_X |f_n - f| < \varepsilon \ \forall n \geq n_0$.

By above $\exists E \ni \mu(E) < \infty$ & $\forall i \in \{1, \cdots n_0\}$, $\int_{E^c} |f_i| < \varepsilon/2$

& $\int_{E^c} |f| < \varepsilon/2$

$\Rightarrow \forall n \le n_0 \longrightarrow$ done

$\forall n \geq n_0$, $\displaystyle\int_{E^c} |f_n| \leq \underbrace{\int_{E^c} |f_n - f|}_{} + \int_{E^c} |f|$

$$\leq \underbrace{\int_X |f_n - f|}_{\varepsilon/2} + \varepsilon/2 = \varepsilon \qquad \text{QED.}$$

Converse: Say $(f_n) \longrightarrow f$ in meas

$$\left.\begin{array}{l} \{f_n\} \text{ is UI } \quad \& \quad \{f_n\} \text{ is tight.} \end{array}\right\} \text{NTS } (f_n) \longrightarrow f \text{ in } L^1.$$

① Assume $f \in L^1$ (You check $\longrightarrow$ not needed).

② Want $\int_X |f_n - f| < \varepsilon$.

$$\int_X |f_n - f| = \underbrace{\int_{\{|f_n - f| \geq \lambda\}}}_{\text{Use U.I. to make this small}} + \underbrace{\int_{\{|f_n - f| < \lambda\}}}_{}$$

$$\underbrace{\int_{\substack{\{|f_n-f|<\lambda\}\cap E\\ \text{(tightness)}}} |f_n-f|}_{\le\ \underbrace{\lambda}_{\lambda<\frac{\varepsilon}{3\mu(E)}}\mu(E)} + \underbrace{\int_{\cancel{\{|f_n-f|<\lambda\}}\cap E^c}}_{<\ \varepsilon\ \text{(by tightness)}}$$

Theorem 8.33 (Vitali)**.** *Let* $(f_n) \in L^1(X)$*. The sequence* (f_n) *is convergent in* L^1 *if and only if*

1. *(1)* (f_n) *converges in measure,*
2. *(2)* (f_n) *is uniformly integrable,*
3. *(3)* (tightness) *there exists* $F \in \Sigma$ *with* $\mu(F) < \infty$ *such that* $\int_{F^c} |f_n| \, d\mu < \varepsilon$ *for all* n*.*

$\forall \varepsilon > 0,$

Proof:

$\{f_n\}$ is U.I. if $\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon \quad \forall n.$

Theorem 8.34. *If* $\lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n| \, d\mu = 0$, *then* (f_n) *is uniformly integrable.*

Theorem 8.35. *If there exists an increasing function* $\varphi\colon [0,\infty) \to [0,\infty)$ *such that* $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$, *and* $\sup_n \int_X \varphi(|f_n|) \, d\mu < \infty$, *then* (f_n) *is uniformly integrable.*

Super linear

Remark 8.36. The hypothesis in both the above theorems are equivalent.

Remark 8.37. If additionally $\boxed{\sup_n \int_X |f_n| \, d\mu < \infty,}$ then the converse of both the above theorems are true.

Rmk: $f_n \in L^1 \Rightarrow \lim_{\lambda\to\infty} \int_{\{|f_n|>\lambda\}} |f_n| \, d\mu \xrightarrow{DCT} 0$

Pf: Pick $\varepsilon > 0$. NTF $\delta > 0$ ∋ $\mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon \quad \forall n$

$\exists \lambda_0$ ∋ $\forall \lambda > \lambda_0$, $\int_{\{|f_n| > \lambda\}} |f_n| < \varepsilon/2$ (by assumption).

Now $\mu(E) < \delta \Rightarrow \int_E |f_n| = \underbrace{\int_{E \cap \{|f_n| > \lambda\}} |f_n|}_{} + \underbrace{\int_{E \cap \{|f_n| \leq \lambda\}} |f_n|}_{}$

$$\leq \frac{\varepsilon}{2} + \delta \cdot \lambda$$

Choose $\delta = \frac{\varepsilon}{2\lambda} \Rightarrow$ QED.

Pf of 8.35: $\frac{\varphi(x)}{x} \xrightarrow{x\to\infty} \infty$, φ inc, & $\sup_n \int_X \varphi(|f_n|)\,d\mu < \infty$

NTS $\{f_n\}$ is U.I.

W.ll show $\lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n|\,d\mu = 0 \Rightarrow$ QED.

Let $\varepsilon>0$: $\Rightarrow \exists\, \lambda_0 \ni \underline{\forall \lambda > \lambda_0}$ $\frac{\varphi(\lambda)}{\lambda} > \frac{1}{\varepsilon} \iff \underline{\lambda \le \varepsilon\varphi(\lambda)}.$

$$\Rightarrow \int_{\{|f_n|>\lambda_0\}} |f_n| \le \varepsilon \int_{\{|f_n|>\lambda_0\}} \varphi(|f_n|) \le \varepsilon \int_X \varphi(|f_n|) \le \varepsilon \sup_n \underbrace{\int_X \varphi(|f_n|)}_{}$$

Proof:

$$\Rightarrow \lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n| = 0 \Rightarrow \text{QED.}$$

Corollary 8.38. *If $(f_n) \to f$ in measure, $\boxed{\mu(X) < \infty}$, and $\sup_n \|f\|_p < \infty$ for any $p > 1$, then $(f_n) \to f$ in L^q for every $\boxed{q \in [1, p)}$.*

Quick check for $q = 1$:

NTS $(f_n) \longrightarrow f$ in L^1.

Vitali: ETS (f_n) is U.I. (have $(f_n) \xrightarrow{\mu} f$ & tightness)

$\varphi(x) = x^p \quad (p > 1)$

$\limsup_n \int_X \varphi(|f_n|) \le \sup_n \int_X |f_n|^p < \infty \Rightarrow (f_n)$ is U.I. $\Rightarrow$ (QED).

9. Signed Measures

9.1. Hanh and Jordan Decomposition Theorems.

(X, Σ) → alg on X.

Definition 9.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
(2) $\mu(\emptyset) = 0$
(3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$.

(Claim: If $\sum_1^\infty \mu(A_i)$ is cgt it must be absolutely cgt)

Example 9.2. Let $f \in L^1(X, \mu)$, and define ν by $\nu(A) = \int_A f\, d\mu$. Then ν is a signed measure, and we write $d\nu = f\, d\mu$.

Example 9.3. If μ, ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is ~~finite~~. a signed measure.

Theorem 9.4 (Jordan Decomposition)**.** *Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.*

IOU.

(Note: μ is a signed measure & $A \subseteq B \not\Rightarrow \mu(A) \leq \mu(B)$.)

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all measurable sets $B \subset A$.

Proposition 9.6. *If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and $\boxed{\mu(B) \leqslant \mu(A)}$.*

Lemma;

Pf: Case ①: $\mu(A) \geq 0 \longrightarrow$ Chose $B = \phi$. $\Rightarrow$ QED.

Case ② $\mu(A) < 0$.

If $\sup\{\mu(E) \mid E \subseteq A\} \leq 0 \Rightarrow A$ is $-$ve, chose $B = A \Rightarrow$ QED.

If $\sup\{\mu(E) \mid E \subseteq A\} = \delta_1 > 0$ find & $E_1 \ni \mu(E_1) \geq \frac{\delta_1}{2} \wedge 1$.

② Let $\delta_2 = \sup\{\mu(E) \mid E \subseteq A - E_1\}$ & find $E_2 \ni \mu(E_2) \geq \frac{\delta_2}{2} \wedge 1$

⋮

ⓝ Let $\delta_n = \sup\{\mu(E) \mid E \subseteq A - \bigcup_1^{n-1} E_k\}$ & find $E_n \ni \mu(E_n) \geq \frac{\delta_n}{2} \wedge 1$

If at any stage $\delta_n \le 0 \longrightarrow$ done: $A - \bigcup_1^{n-1} E_k$ is $-$ve $\Rightarrow$ QED.

__Claim:__ $\sum_1^\infty \delta_i < \infty$. ($\because$ ~~$\mu(A) = \mu(\underbrace{A - \bigcup_1^\infty E_k}_{B} \cup \bigcup_1^\infty E_k)$~~)

Let $B = A - \bigcup_1^\infty E_k$. Then $\underline{\mu(A)} = \mu(B \cup \bigcup_1^\infty E_k) = \underline{\mu(B)} + \underbrace{\sum_1^\infty \mu(E_k)}_{\ge 0}$

$\because \mu(A) < \infty \Rightarrow \sum_1^\infty \mu(E_k) < \infty$.

__Claim 2:__ B is $-$ve & $\mu(B) \le \mu(A)$ ($\Rightarrow$ QED).

NTS B is $-$ve.

Note $\sum \mu(E_{\cancel{m}k}) < \infty \Rightarrow \underline{\sum \delta_k < \infty}$.

Also, $E \subseteq \cancel{A} B \Rightarrow E \subseteq A - \bigcup_{1}^{n-1} E_k \Rightarrow \mu(E) \leq \delta_n \xrightarrow{n \to \infty} 0$

$\Rightarrow \mu(E) \leq 0$.

QED.

9. Signed Measures

9.1. Hanh and Jordan Decomposition Theorems.

Definition 9.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
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Theorem 9.4 (Jordan Decomposition)**.** *Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.*

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all measurable sets $B \subseteq A$.

~~Proposition~~ 9.6. *If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and $\mu(B) \leqslant \mu(A)$.*

Lemma

(Last time)

Theorem 9.7 (Hanh decomposition)**.** *If μ is a signed measure on X, then $X = P \cup N$ where P is positive and N is negative.*

Remark 9.8. The decomposition is unique up to null sets.

Pf: $X = P' \cup N' = P \cup N$ $\quad$ (P', P +ve $\quad N', N$ −ve)

$$\Rightarrow P = \underbrace{(P \cap P')}_{+ve} \cup \underbrace{(P \cap N')}_{\text{both +ve \& −ve}}$$

⇒ all subsets of $P \cap N'$ are meas 0

$\Rightarrow P = P' \cup$ null set.

Pf of Existence: ① WL assume $-\infty \notin$ range(μ).

② Let $\alpha = \inf \{\mu(E) \mid E \subseteq X\}$. $\quad$ (α could be $-\infty$)

③ Chose $(\alpha_n) \ni (\alpha_n) \longrightarrow \alpha$ & $\alpha < \alpha_{n+1} < \alpha_n$.

$\Rightarrow \forall n, \ \exists A_n \ni \alpha \leq \mu(A_n) < \alpha_n$

$\Rightarrow$ By lemma $\exists B_n$ negative $\ni$ $B_n \subseteq A_n$.

& $\underline{\mu(B_n) \leq \mu(A_n)}$

$\Rightarrow \alpha \leq \mu(B_n) < \alpha_n$

④ Let $N = \bigcup_{1}^{\infty} B_n$. Clearly $\alpha \leq \mu(N) < \alpha_n \ \forall n$

& N is $-$ve

$\Rightarrow \mu(N) = \alpha \qquad (\Rightarrow \alpha > -\infty)$

⑤ NTS $P = N^c$ is +ve.

Let $E \subseteq P$. NTS $\mu(E) \geq 0$.

If $\mu(E) < 0 \Rightarrow \mu(E \cup N) = \mu(E) + \mu(N) < \alpha$

($\because$ α is finite)

Contradiction

$\Rightarrow \mu(E) \geq 0 \Rightarrow P$ is +ve $\Rightarrow$ QED.

Definition 9.9. We say two positive measures μ, ν are $\underbrace{\textit{mutually singular}}_{\mu \perp \nu}$ if there exists $C \subseteq X$ such that for every $A \in \Sigma$ we have $\mu(A \cap C) = \nu(A \cap C^c) = 0$.

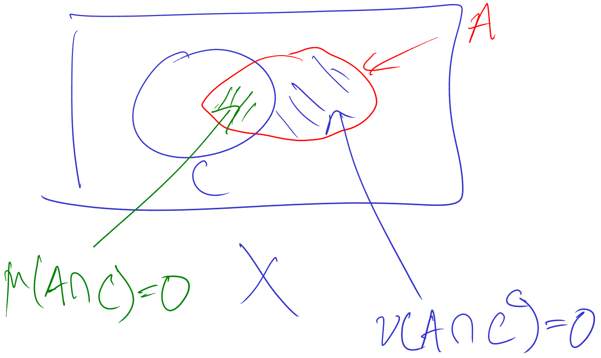
Proof of Theorem 9.4

If μ is a signed meas

then $\exists!$ μ^+ & μ^- $\ni$ $\mu^+ \perp \mu^-$

$\ni$ $\mu = \mu^+ - \mu^-$

(μ^+ & μ^- are +ve meas)



Pf: $X = P \cup N$ by Hahn. Set $\left.\begin{array}{l} \mu^+(A) = \mu(A \cap P) \\ \mu^-(A) = -\mu(A \cap N). \end{array}\right\} \Rightarrow$ existence.

Uniqueness $\rightarrow$ $\mu = \mu^+ - \mu^- = \nu^+ - \nu^-$, $\mu^\pm, \nu^\pm \geq 0$, $\mu^+ \perp \mu^-$

$\nu^+ \perp \nu^-$

$$\mu^+ \perp \mu^- \Rightarrow \exists\, C \ni \mu^+(C) = 0 = \mu^-(C^c)$$

$$\nu^+ \perp \nu^- \Rightarrow \exists\, D \ni \nu^+(D) = 0 = \nu^-(D^c).$$

$$\Rightarrow X = \underbrace{C}_{\substack{-\text{ve} \\ \text{wrt } \mu}} \cup \underbrace{C^c}_{\substack{+\text{ve} \\ \text{wrt } \mu}} = \underbrace{D}_{\substack{-\text{ve} \\ \text{wrt } \mu}} \cup \underbrace{D^c}_{\substack{+\text{ve} \\ \text{wrt } \mu.}}$$

Uniqueness of hahn $\Rightarrow$ QED.

Definition 9.10. Let μ be a signed measure with Jordan decomposition $\boxed{\mu = \mu^+ - \mu^-}$. Define the $\boxed{\text{variation of } \mu}$ to be the (positive) measure $|\mu| \overset{\text{def}}{=} \mu^+ + \mu^-$.

Definition 9.11. Define the *total variation* of μ by $\|\mu\| = |\mu|(X)$. $\in [0, \infty]$

Proposition 9.12. *Let $\mathcal{M}$ be the set of all finite signed measures on X. Then $\mathcal{M}$ is a Banach space under the total variation norm.*

WTS ① $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ } ← std def done.

& ② Completeness

Q: $x \in \mathbb{R}$. $\delta_x = \delta$ mass at x: $\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

Q: Does $\delta_{1/n} \longrightarrow \delta_0$ in $\mathcal{M}$? (No: $\|\delta_{1/n} - \delta_0\| = 2$)

Well

9.2. Absolute Continuity.

Definition 9.13. Let μ, ν be two the measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g \, d\mu$. (Notation: Say $d\nu = g \, d\mu$.) ← $\boxed{\int f \, d\nu = \int f g \, d\mu}$

Theorem 9.15 (Radon-Nikodym). *If μ, ν are two σ-finite positive measures with $\nu \ll \mu$, then there exists a measurable function g such that $0 \leqslant g < \infty$ almost everywhere and $d\nu = g \, d\mu$.* → $\nu(A) = \int_A g \, d\mu$

Pf:

Case I: μ & ν are finite.

Let $\mathcal{F} = \{ f \geq 0 \mid \forall A \in \Sigma, \int_A f \, d\mu \leq \nu(A) \}$

Q: $\mathcal{F} = \emptyset$? (No → $f = 0 \in \mathcal{F}$)

Guess: g = "largest element of $\mathcal{F}$" ~~$g(x) = \sup_{f \in \mathcal{F}} f(x)$~~ ← Won't EVER Work.

Let $\alpha = \sup_{f \in \mathcal{F}} \int_X f \, d\mu \leq \nu(\cancel{A}X) < \infty$.

$\Rightarrow \forall n \ \exists f_n \in \mathcal{F} \ni \int_X f_n \, d\mu \geq \alpha - \frac{1}{n}$

(Replace f_n with $\max\{f_n, f_{n-1}\}$ & assume WL (f_n) is inc)

Note: $f_1, f_2 \in \mathcal{F} \Rightarrow f_1 \vee f_2 \in \mathcal{F}$ $\quad (a \vee b = \max(a,b))$

$$\left(\because \int_A (f_1 \vee f_2)\, d\mu = \int_{A \cap \{f_1 > f_2\}} f_1 \, d\mu + \int_{A \cap \{f_1 \leq f_2\}} f_2 \, d\mu \leq \nu(A \cap \{f_1 > f_2\}) + \nu(A \cap \{f_1 \leq f_2\}) = \nu(A) \text{ QED}\right).$$

Au $\Rightarrow \int_X f_n \, d\mu \geq \alpha - \frac{1}{n}$ & $f_{n+1} \geq f_n$ & $f_n \in \mathcal{F}$.

Set $g = \lim f_n$ (must exist).

Next time: $\int_A g \, d\mu = \int_A \nu(A) \quad \forall A \in \Sigma$.

9.2. Absolute Continuity.

Definition 9.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g \, d\mu$. (Notation: Say $d\nu = g \, d\mu$.)

Theorem 9.15 (Radon-Nikodym)**.** *If μ, ν are two positive measures such that ν is σ-finite and $\nu \ll \mu$, then there exists a unique measurable function g such that $0 \leqslant g < \infty$ almost everywhere and $d\nu = g \, d\mu$.*

μ, ν

Last time: ① Case I: $\nu(X) < \infty$. $(\& \ \mu(X) < \infty)$

② $\mathcal{F} = \{ f \mid f \geq 0 \ \& \ \forall A, \ \int_A f \, d\mu \leq \nu(A) \}$

③ Note if $f_1, f \in \mathcal{F} \Rightarrow f_1 \vee f_2 \in \mathcal{F}$.

④ $\Rightarrow \exists$ a seq $(f_n) \ni \int_X f_n \, d\mu \longrightarrow \sup_{f \in \mathcal{F}} \int_X f \, d\mu$ (Exercise: $< \nu(X) < \infty$)

by ③ can ensure $f_n \leq f_{n+1}$

⑤ Let $g = \lim_{n \to \infty} f_n$.

$$\boxed{\underline{\text{Claim}} \quad d\nu = g\, d\mu}$$

⑥ Let $d\lambda = d\nu - g\, d\mu$ (i.e. $\lambda(A) = \nu(A) - \int_A g\, d\mu$.)

Note: $g \in \mathcal{F}$ (M.C.) $\Rightarrow$ λ is a +ve meas.

⑦ NTS $\lambda = 0$. Will show $\forall \varepsilon > 0$, $\lambda \leq \varepsilon \mu$. ($\Rightarrow$ <u>QED</u>)

Note $\lambda - \varepsilon\mu$ is a signed measure. Let $X = P \cup N$ be the Hahn decomposition of $\lambda - \varepsilon\mu$.

Claim: $g + \varepsilon \mathbb{1}_P \in \mathcal{F}$.

↳ Pf: NTS $\forall A$, $\int_A (g + \varepsilon \mathbb{1}_P)\, d\mu \le \nu(A)$

$$\int_A (g + \varepsilon \mathbb{1}_P)\, d\mu = \nu(A) - \lambda(A) + \varepsilon \mu(A \cap P)$$

$$= \nu(A) - \underbrace{\lambda(A \cap N)}_{\ge 0} - \underbrace{\left(\lambda(A \cap P) - \varepsilon \mu(A \cap P)\right)}_{\ge 0 \ (\because P \text{ is } +\text{ve for } \lambda - \varepsilon\mu)}$$

$$\le \nu(A)$$

⇒ Claim.

$\Rightarrow g + \varepsilon \mathbb{1}_P \in \mathcal{F} \Rightarrow \mu(P) = 0 \overset{a.c.}{\Longrightarrow} \nu(P) = 0$

$\int_X g\, d\mu = \sup_{f \in \mathcal{F}} \int_X f\, d\mu$

$$\Rightarrow \lambda(P) = 0 \Rightarrow (\lambda - \varepsilon\mu)(P) = 0$$

$\Rightarrow \lambda - \varepsilon\mu$ is a $-$ve meas

$$\Rightarrow \lambda \leq \varepsilon\mu. \Rightarrow \text{QED}.$$

Uniqueness: If $d\nu = g\,d\mu = h\,d\mu \Rightarrow g = h$ a.e.

Pf: $\forall A, \int_A g\,d\mu = \int_A h\,d\mu \Rightarrow \int_A (g-h)\,d\mu = 0 \ \forall A.$

Choose $A = \{g-h > 0\}. \Rightarrow \int_{\{g>h\}} (g-h)\,d\mu = 0 \Rightarrow \mu\{g>h\} = 0$

lly $\mu\{g<h\} = 0 \Rightarrow g = h$ a.e.

Case II: Write $X = \bigcup F_n$, $\mu(F_n) < \infty$, $\nu(F_n) < \infty$.

WL. assume $F_n \subseteq F_{n+1}$.

By Case I, $\exists\, g_n$ s.t. $\forall A$, $\nu(A \cap F_n) = \int_{A \cap F_n} g_n \, d\mu$

By uniqueness $g_{n+1}\Big|_{F_n} = g_n$.

Set $g = \lim g_n$ (is an inc lim).

$$\nu(A) = \lim_{n\to\infty} \nu(A \cap F_n) = \lim_{n\to\infty} \int_{A\cap F_n} g \, d\mu = \lim_{n\to\infty} \int_A \mathbb{1}_{F_n} g \, d\mu \overset{MC}{=} \int_A g \, d\mu.$$

QED.

Theorem 9.16. *Let μ, ν be positive measures such that* ~~μ,~~ *ν is σ-finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.*

Pf: Case I : ~~μ~~, ν finite

Let $\mathcal{N} = \{ A \mid \mu(A) = 0 \}$ &

Consider $\sup \{ \nu(A) \mid A \in \mathcal{N} \}$, & find N_k ∋ $\nu(N_k) \xrightarrow{k \to \infty} \sup_{A \in \mathcal{N}} \nu(A)$

Let $N = \bigcup_{k=1}^{\infty} N_k$. & $\nu_s(A) = \nu(A \cap N)$, $\nu_{ac}(A) = \nu(A \cap N^c)$.

<u>Claim</u> : ① $\nu_s \perp \mu$.

② $\nu_{ac} \ll \mu$.

Pf of ① : $\nu_s(N^c)=0$ & $\mu(N)=0$ $\Rightarrow$ ①.

Pf of ② : NTS $\mu(A)=0 \Rightarrow \nu_{ac}(A)=0$

i.e. NTS $\mu(A)=0 \Rightarrow \nu(A\cap N^c)=0$

$$\nu(N) \le \nu(A\cup N) \le \nu(N) \quad \left(\because \nu(N)=\sup_{\mu(B)=0}\nu(B) \ \& \ \mu(A\cup N)=0\right)$$

$(\because \nu$ is finite$)$ $\Rightarrow \nu(A-N)=0 \Rightarrow \nu(A\cap N^c)=0$

QED.

Theorem 9.16. *Let μ, ν be positive measures such that ν is σ-finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.*

Last time → did the proof of existence when ν is finite.

Uniqueness: Say $\nu = \nu_{ac} + \nu_s = \tilde{\nu}_{ac} + \tilde{\nu}_s$

$$\left.\begin{array}{l} \exists N \ni \mu(N) = 0 \ \& \ \nu_s(N^c) = 0 \\ \exists \tilde{N} \ni \mu(\tilde{N}) = 0 \ \& \ \tilde{\nu}_s(\tilde{N}^c) = 0 \end{array}\right\} \begin{array}{l} \hat{N} = N \cup \tilde{N} \\ (\mu(\hat{N}) = 0 \end{array}$$

Pick any $A \in \Sigma$. $(\nu_s - \tilde{\nu}_s)(A) = (\nu_s - \tilde{\nu}_s)(A \cap \hat{N}) = 0$

also $(\nu_s - \tilde{\nu}_s)(A) = (\nu_{ac} - \tilde{\nu}_{AC})(A) = (\nu_{ae} - \tilde{\nu}_{ae})(\underbrace{A \cap \hat{N}}_{\mu(A \cap \hat{N}) = 0}) = 0$

$\Rightarrow$ $\nu_s(A) = \tilde{\nu}_s(A)$ & $\nu_{ac}(A) = \tilde{\nu}_{ac}(A)$ $\Rightarrow$ uniq QED.

Case II: ν is σ-finite.

Write $X = \bigcup_{1}^{\infty} F_n$, $F_n \subseteq F_{n+1}$ & $\nu(F_n) < \infty$.

$\forall n$, let $\nu^{(n)}(A) = \nu(A \cap F_n)$. Case I $\Rightarrow$ can write $\nu^{(n)} = \nu^{(n)}_{ac} + \nu^{(n)}_{s}$.

also $\exists$ $N_n \subseteq F_n$ s.t. $\mu(N_n) = 0$, & $\nu_s^{(n)}(A) = \nu^{(n)}_{\cancel{e}}(A \cap N_n)$

$= \nu(A \cap N_n \cap F_n)$.

Now set $N = \bigcup_{1}^{\infty} N_n$, $\nu_{ac}(A) = \nu(A \cap N^c)$ & $\nu_s(A) = \nu(A \cap N)$.

Clearly $\nu_s \perp \mu$. $\left(\because \mu(N) = 0 \right)$

NTS $\nu_{ac} \ll \mu$: Say $\mu(A) = 0$. Then $\nu_{ac}(A) \underset{\uparrow}{=} \lim \nu_{\underline{\underline{ac}}}^{(n)}(A) = 0$

QED

Corollary 9.17. *Let μ be a positive measure, and $\underline{\nu}$ be a finite signed measure. There exists a unique pair of signed measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$ and $\nu = \underline{\nu_{ac}} + \underline{\nu_s}$.*

Pf: $\nu = \nu^+ - \nu^-$ & write $\nu^\pm = \nu^\pm_{ac} + \nu^\pm_s$

QED.

(Radon Nicodym)

Corollary 9.18. *Let μ, ν be σ-finite positive measures. There exists a unique positive measure ν_s and nonnegative measurable function g such that $\mu \perp \nu_s$ and $d\nu = d\nu_s + g\, d\mu$.*

Pf: Know $\nu = \nu_{ac} + \nu_s$. & by RN know $\exists g \ni d\nu_{ac} = g\, d\mu$.

QED.

9.3. Dual of L^p.

Proposition 9.19. *Let U, V be Banach spaces, and $T\colon U \to V$ be linear. Then T is continuous if and only if there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.*

bdd.

Pf: Say T is cts. $\Rightarrow$ T is cts at 0 $\Rightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ + $\|u - 0\|_U < \delta \Rightarrow$ $\|Tu - \underset{=0}{T0}\|_V < \varepsilon$

$\Rightarrow \|u\|_U < \delta \Rightarrow \|Tu\|_V < \varepsilon$.

$\Rightarrow \forall u \in U$, $\left\|\frac{\delta u}{2\|u\|}\right\| = \frac{\delta}{2}$. $\Rightarrow \left\|T\left(\frac{\delta u}{2\|u\|}\right)\right\| < \varepsilon$

$\Rightarrow$ (linearity) $\|Tu\| \leq \frac{2\varepsilon\|u\|}{\delta}$

Chose $c = \frac{2\varepsilon}{\delta}$ $\Rightarrow$ QED.

Conversely: Assume $\|Tu\|_V \leq c\|u\|_U$. NTS T is cts.

Note $|Tu_1 - Tu_2\|_V \overset{\text{linear}}{=} |T(u_1-u_2)\| \leq c\|u_1-u_2\|_V$.

$\Rightarrow$ T is Lipschitz $\Rightarrow$ cts

QED

Definition 9.20. We say $T\colon U \to V$ is a bounded linear transformation if T is linear and there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.

← u^* is called a bdd linear functional

Definition 9.21. The *dual* of U is defined by $U^* = \{u^* \mid u^*\colon U \to \mathbb{R}$ is bounded and linear.$\}$ Define a norm on U^* by

$$\|u^*\|_{U^*} \overset{\text{def}}{=} \sup_{u\in U-0} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U=1} \cancel{\frac{1}{\|u\|_U}} u^*(u) = \sup_{\|u\|_U=1} \cancel{\frac{1}{\|u\|_U}} |u^*(u)| .$$

Proposition 9.22. *The dual of a Banach space is a Banach space.*

$$\mathcal{L}(U,V) = \{ T \mid T\colon U \to V \; \cancel{\}} \; T \text{ bdd linear} \}$$

$$\|T\|_{\mathcal{L}(U,V)} = \sup_{u \neq 0} \frac{\|Tu\|_V}{\|u\|_U}$$

(You check)

$p\in[1,\infty]$

Proposition 9.23. *Let $1/p+1/q=1$, $g\in L^q(X)$. Define $T_g\colon L^p\to\mathbb{R}$ by $T_g f=\int_X fg\,d\mu$. Then $T_g\in(L^p)^*$.* ✓

Proposition 9.24. *The map $g\mapsto T_g$ is a bounded linear map from $L^q\to(L^p)^*$.*

$T_{(g_1+g_2)}f = \int (g_1+g_2)f$

$= T_{g_1}f + T_{g_2}f$

Pf: Clearly $T_g(f_1+f_2) = T_g f_1 + \cancel{T_g}\, T_g f_2$.

also $|T_g f| = \left|\int_X fg\,d\mu\right| \overset{\text{Holder}}{\le} \|f\|_p \boxed{\|g\|_q} \Rightarrow$ cts.

c.

ℓ

$\Rightarrow T_g \cancel{f} \in (L^p(X))^*$.

Also note $\|T_g\|_{(L^p)^*} \le \|g\|_q$ (Claim: Duality from before $\Rightarrow$

$\|T_g\|_{(L^p)^*} = \|g\|_q$).

Theorem 9.25. *Let (X, Σ, μ) be a σ-finite measure space, $p \in [1, \infty)$, $1/p + 1/q = 1$. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.*

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

requires work.

(Next time.)

Pr : $(\Omega, \boxed{\mathcal{F}}, P)$

$\uparrow P(\Omega) = 1.$

R.V. $\rightarrow$ $\boxed{X} : \Omega \longrightarrow \mathbb{R}$ $\mathcal{F}$-meas is a R.V.

RV, Y $\longleftarrow$ Observe Y. $\longrightarrow$ $\overbrace{\text{What events can you deduce}}$ the fr of?

$\swarrow$

$$\underline{\sigma(Y)} = \sigma \text{ alg gen by } \{ \underline{Y^{-1}(U)} \mid U \subseteq \mathbb{R} \text{ open} \}$$

$$E(X \mid \underline{Y}) = E(\underline{X \mid}\ \underline{\sigma(Y)})$$

on

$$\sigma\left(Y^{-1}(U) \mid U \subseteq \mathbb{R} \text{ is open}\right)$$

Ex $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \mid \mathcal{F}) = E(\mathbb{1}_A \mid \mathcal{F})$$

Theorem 9.25. *Let (X, Σ, μ) be a σ-finite measure space, $p \in [1, \infty)$, $1/p + 1/q = 1$. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.*

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

Pf: $g \in L^q$. $T_g \in (L^p)^*$ $\quad T_g f = \int_X f g \, d\mu$.

$$\|T_g\|_{(L^p)^*} = \sup_{f \in L^p, \|f\|_p = 1} |T_g(f)| = \sup_{f \in L^p, \|f\|_p = 1} \left| \int_X f g \, d\mu \right|$$

Duality Lemma $\quad \|g\|_{L^q}$

$\Rightarrow g \to T_g$ is an isom from $L^q \to (L^p)^*$

NTS $g \longrightarrow T_g$ is surjective. i.e. If $\Lambda \in (L^p)^*$, NTS $\exists g \in L^q \ni T_g = \Lambda$

Case I: Say μ is finite.

Define $\nu(A) = \Lambda(\mathbb{1}_A)$.

Claim: ν is a measure!

Pf: Say $A_1, A_2, \ldots$ ctably many disj sets

<u>Claim</u>: $\mathbb{1}_{\bigcup_1^N A_n} \xrightarrow{L^p} \mathbb{1}_{\bigcup_1^\infty A_n}$.

Pf: $\|\mathbb{1}_{\bigcup^\infty A_n} - \mathbb{1}_{\bigcup_1^N A_n}\|_p^p = \int_X \left(\sum_{N+1}^\infty \mathbb{1}_{A_n}\right)^p \overset{\text{disj}}{=} \sum_{N+1}^\infty \mu(A_n)^{*} \xrightarrow[N\to\infty]{} 0$

$$\Rightarrow \nu\Big(\bigcup_{1}^{\infty} A_n\Big) = \Lambda\Big(\mathbb{1}_{\bigcup_{1}^{\infty} A_n}\Big) \overset{\text{claim \& continuity of } \Lambda}{=\!=\!=} \lim_{N\to\infty} \Lambda\Big(\mathbb{1}_{\bigcup_{1}^{N} A_n}\Big) \quad (\because \mu(X) < \infty).$$

$$= \lim_{N\to\infty} \sum_{1}^{N} \nu(A_n) \quad (\text{linearity of } \Lambda)$$

$$= \sum_{1}^{\infty} \nu(A_n).$$

⇒ ν is a signed measure

Claim: $\nu \ll \mu$. (Pf: $\mu(A) = 0 \Rightarrow \nu(A) = \Lambda(\mathbb{1}_A) = \Lambda(0) = 0$)

($\because \mathbb{1}_A = 0$ a.e.)

$\Rightarrow$ By R.N. $\exists g$ ~~meas~~ int. + $\nu(A) = \int_A g \, d\mu$.

Claim: ① $g \in L^q$. Claim ②: $\Lambda(s) = \int_X s g \, d\mu$ $\forall s$ simple.

Claim ③: $\Lambda(f) = \int_X f g \, d\mu$ $\forall f \in L^p$.

Note Pf of 2: $\Lambda(\mathbb{1}_A) \overset{\text{def}}{=} \nu(A) = \int_A g \, d\mu$ & linearity $\Rightarrow$ Claim ② QED.

Pf of Claim ①: Know $\|g\|_{L^q} = \sup_{\|f\|_p} \int_X fg \, d\mu \underset{\substack{\uparrow \\ \text{you check}}}{=} \sup_{\substack{s \text{ simple} \\ \|s\|_p = 1}} \int_X sg \, d\mu$

$$= \sup_{\substack{s \text{ simple} \\ \|s\|_p = 1}} \Lambda(s) \quad (\text{by Claim } 2)$$

$$\leq \|\Lambda\|_{(L^p)^*} < \infty \Rightarrow g \in L^q.$$

D.C. ⇒ Claim 3 ⇒ QED.

Case II : $X = \bigcup F_n$, $F_n \subseteq F_{n+1}$ & $\mu(F_n) < \infty$.

Note the fn g from case I is unique $\Lambda(\mathbb{1}_{F_{n+1}} f) = \int_{F_{n+1}} f \, g_{n+1}$

$\Rightarrow \forall n \ \exists g_n \in L^q$ + $\Lambda(\mathbb{1}_{F_n} f) = \int_X g_n f \, \mathbb{1}_{F_n} \, d\mu$.

By uniqueness, $g_{n+1} \mathbb{1}_{F_n} = g_n \mathbb{1}_{F_n}$. Let $g = \lim_{n \to \infty} g_n$ (must exist)

Claim: $g \in L^q$. (Pf: $\int_X |g|^q \underset{MC}{=} \lim \int_{F_n} |g|^q = \lim \int_{F_n} |g_n|^q$

$$= \lim_{n\to\infty} \left\| \Lambda|_{F_n} \right\|_{(L^p(F_n))^*} \leq \|\Lambda\|_{(L^p(X))^*} \Big)$$

Now $\forall f \in L^p$,

$$\int_X fg \underset{\substack{\uparrow \\ DC \\ (\because\, fg \in L^1)}}{=} \lim_{n\to\infty} \int_{F_n} fg = \lim_{n\to\infty} \Lambda(\mathbb{1}_{F_n} f) = \Lambda(f)$$

QED.

9.4. Riesz Representation Theorem.

Theorem 9.28 (Riesz Representation Theorem)**.** *Let X be a compact metric space, and $\mathcal{M}$ be the set of all finite signed measures on X. Define $\Lambda\colon \mathcal{M} \to C(X)^*$ by $\Lambda_\mu(f) = \int_X f\,d\mu$ for all $\mu \in \mathcal{M}$ and $f \in C(X)$. Then Λ is a bijective linear isometry.*

Remark 9.29. In particular, for every $I \in C(X)^*$, there exists a unique finite regular Borel measure μ such that $I(f) = \int_X f\,d\mu$ for every $f \in C(X)$.

μ a finite signed measure on X

$f \in C(X)$.

$$T_\mu(f) = \int_X f \, d\mu$$

$$|T_\mu(f)| \leq \|f\|_\infty \underbrace{|\mu|(X)}_{\|\mu\|}$$

$$(L^p)^* = \{ \Lambda \mid \Lambda : L^p \longrightarrow \mathbb{R} \text{ is } \underbrace{\text{bdd}}_{\text{cts}} \text{ \& linear} \}$$

$g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $T_g \in (L^p)^*$ def by

$$T_g(f) = \int_X fg \, d\mu$$

10. Product measures

← rectagles

Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\ B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 10.1. *Let μ, ν be two σ-finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.*

Theorem 10.2 (Tonelli)**.** *Let $f \colon X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$-measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,*

$$\int_{X \times Y} f(x,y)\, d\pi(x,y) = \int_{x \in X} \Big(\int_{y \in Y} f(x,y)\, d\nu(y) \Big)\, d\mu(x) = \int_{y \in Y} \Big(\int_{x \in X} f(x,y)\, d\mu(x) \Big)\, d\nu(y)\,. \tag{10.1}$$

Theorem 10.3 (Fubini)**.** *If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover,*(10.1) *holds.*

10. Product measures

Rectangles. Product σ-alg.

Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\ B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 10.1. *Let μ, ν be two σ-finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.*

Theorem 10.2 (Tonelli). *Let $f\colon X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$-measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,*

Iterated Integrals

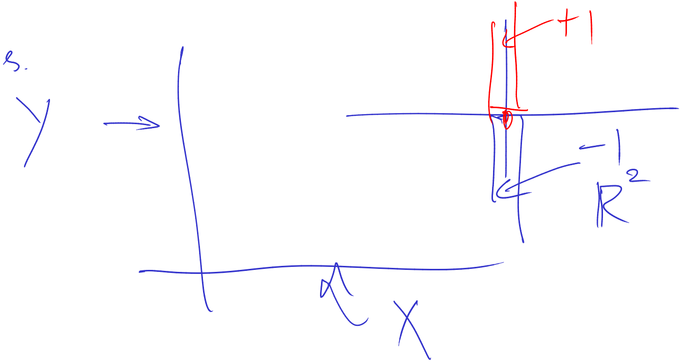
$$\int_{X \times Y} f(x,y)\, d\pi(x,y) = \int_{x \in X} \left(\int_{y \in Y} f(x,y)\, d\nu(y) \right) d\mu(x) = \int_{y \in Y} \left(\int_{x \in X} f(x,y)\, d\mu(x) \right) d\nu(y)\,. \tag{10.1}$$

Theorem 10.3 (Fubini). *If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover,(10.1) holds.*

$f: X \times Y \longrightarrow \mathbb{R}$. $\quad \Sigma \otimes \tau$ meas.

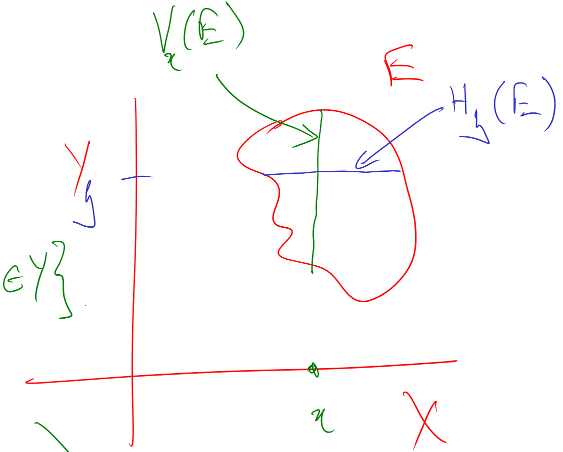
$\forall\ x_0 \in X \qquad y \longmapsto f(x_0, y)$.

$\forall\ y_0 \in Y \qquad x \longmapsto f(x, y_0)$



Lemma 10.4. *For every $E \subseteq X \times Y$, $x \in X$, $y \in Y$ define the horizontal and vertical slices of E by $H_y(E) = \{x \in X \mid (x, y) \in E\}$ and $V_x(E) = \{y \in Y \mid (x, y) \in E\}$.*

1. *For every $x \in X$, $y \in Y$ we have $H_y(E) \in \Sigma$ and $V_x(E) \in \tau$.*
2. *The functions $\underbrace{x \mapsto \nu(V_x(E))}_{\Sigma\text{- meas}}$ and $\underbrace{y \mapsto \mu(H_y(E))}_{\tau\text{- meas.}}$ are measurable.*



Pf: ① $\Lambda = \{E \in \Sigma \otimes \tau \mid H_y(E) \in \Sigma \ \forall y \in Y\}$

Claim: Λ is a σ-alg.

$\left(\text{Pf: } H_y\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} H_y(E_i)\right)$ $\Rightarrow$ QED ① of Lemma.

NTS ②: I.e. NTS. the fn $y \mapsto \mu(H_y(E))$ is τ-meas.

Case I: μ & ν are finite

Pf: $\Lambda = \{E \in \Sigma \otimes \tau \mid$ the fn $y \longmapsto \mu(H_y(E))$ is τ meas$\}$.

Dynkin Systems: ① $\Lambda \supseteq \Sigma \times \tau$ (rectangles) which is a π-sys.

② $E, F \in \Lambda$, $E \subseteq F$, then $F - E \in \Lambda$

(Pf: $$\mu(H_y(F-E)) = \underbrace{\mu(H_y(F))} - \underbrace{\mu(H_y(E))}$$ ($\because \mu, \nu$ are finite).

meas wrt y ($\because E, F \in \Lambda$)

$\Rightarrow y \longmapsto \mu(H_y(F-E))$ is τ meas

$\Rightarrow F - E \in \Lambda.)$

③ $E_n \in \Lambda$, $E_n \subseteq E_{n+1}$.

$$\mu\left(H_y\left(\bigcup_{1}^{\infty} E_n\right)\right) = \lim_{n\to\infty} \underbrace{\mu(H_y(E_n))}_{\tau\text{- meas fns of } y}$$

$$\Rightarrow \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{is a } \tau \text{ meas fn of } y}$$

$$\Rightarrow \bigcup_{1}^{\infty} E_n \in \Lambda.$$

$\therefore$ Λ is a λ-sys & ~~λ~~ $\Lambda \supseteq \Sigma \times \tau \Rightarrow \Lambda \supseteq \sigma(\Sigma \times \tau) = \Sigma \otimes \tau$

QED.

Case II: μ, ν σ-finite: $X = \bigcup F_n$, $Y = \bigcup E_n$.

$\mu(F_n) < \infty$, ~~μ~~ $\nu(E_n) < \infty$, $F_n \subseteq F_{n+1}$, $E_n \subseteq E_{n+1}$

$$\mu(H_y(A)) = \lim_{n\to\infty} \underbrace{\mu(H_y(A \cap (E_n \times F_n)))}_{\text{by case 1 are all } \tau\text{-meas fns.}}$$

$\Rightarrow y \to \mu(H_y(A))$ is also τ-meas.

QED.

Proof of Theorem 10.1 . NTS $\exists!$ meas $\pi \ni \underline{\pi(A \times B)} = \mu(A)\, \nu(B)$.

① Uniqueness $\longrightarrow$ Done before

π_1 & π_2 are 2 product measure

Know $\pi_1 = \pi_2$ on $\Sigma \times \tau$ (π - system)

Know μ & ν are σ - finite

$\Longrightarrow \pi_1 = \pi_2$ on $\sigma(\Sigma \times \tau) = \Sigma \otimes \tau$.

② IOU Existence.

Let $\pi(E) = \int_{y \in Y} \mu(H_y(E))\, d\nu(y)$

(integral is defined $\because$ $y \to \mu(H_y(E))$ is τ - meas & ≥ 0)

ⓐ Is π a measure?

Say $E_n \in \Sigma \otimes \tau$, $E_n \cap E_m = \phi$ if $n \neq m$.

$$\pi\left(\bigcup_1^\infty E_n\right) = \int_{y \in Y} \mu\left(H_y\left(\bigcup_1^\infty E_n\right)\right) \, d\nu(y)$$

$$= \int_{y \in Y} \sum_1^\infty \mu(H_y(E_n)) \, d\nu(y) \overset{MC}{=} \sum_1^\infty \int_{y \in Y} \mu(H_y(E_n)) \, d\nu(y)$$

$$= \sum_{1}^{\infty} \pi(E_n) \qquad \Rightarrow \pi \text{ is a meas.}$$

② $$\pi(A \times B) = \int_{y \in Y} \mu\big(H_y(A \times B)\big)\, d\nu(y)$$

$$= \int_{y \in Y} \mathbb{1}_B(y)\, \mu(A) = \nu(B)\, \mu(A) \qquad \text{QED.}$$

$X \times Y$: $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$

$f : X \times Y \longrightarrow \mathbb{R}$ $\boxed{\mu \,\&\, \nu \text{ are } \sigma\text{-finite}}$

① $\exists !\ \pi$ on $\Sigma \otimes \tau \ni \pi(A \times B) = \mu(A)\,\nu(B)$ (Done)

② $\forall f : X \times Y \to \underline{\underline{[0,\infty)}}$: $\displaystyle\int_{X\times Y} f \, d\pi = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$ ⊛

(Tonelli)

③ (Fubini) ⊛ holds if $f \in \mathcal{L}^1(X \times Y, \Sigma \otimes \tau)$.

Proof of Theorem 10.2 (Tonelli)

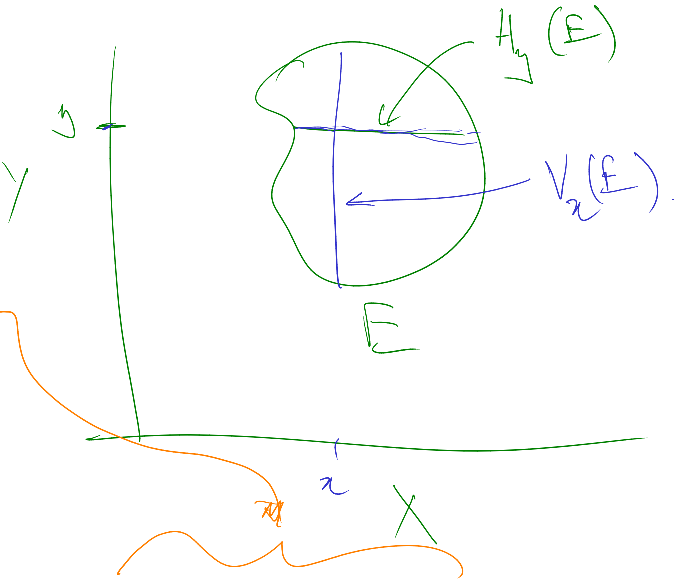
Recall

Last time

$$\to \pi(E) \overset{①}{=} \int_{y\in Y} \mu(H_y(E))\, d\nu(y)$$

Today (uniqueness of π)

$$\overset{②}{=} \int_{x\in X} \nu(V_x E)\, d\mu(x)$$



$$\Rightarrow \pi(E) = \int_{X\times Y} 1_E \, d\cancel{\mu}\pi \overset{①}{=} \int_{y\in Y}\left(\int_{x\in X} 1_E(x,y)\, d\mu(x)\right) d\nu(y)$$

$$\overset{②}{=} \int_{x\in X} \left(\int_{y\in Y} \mathbb{1}_{E}(x,y)\, d\nu(y) \right) d\mu(x).$$

⇒ Tonelli is true for indicator fns.

⇒ Tonelli is true for simple fns (linearity)

⇒ Tonelli is true for +ve fns (Monotone Conv).

QED.

Proof of Theorem 10.3 (Fubini).

$$f \in \mathcal{L}^1(X \times Y) \Rightarrow \int_{X\times Y} |f| \, d\pi \overset{\text{Tonelli}}{=} \int_X \left(\int_Y |f(x,y)| \, d\nu(y) \right) d\mu(x) < \infty$$

$$\Rightarrow \tilde{\forall} x \in X, \ \int_Y |f(x,y)| \, d\nu(y) < \infty \Rightarrow \tilde{\forall} x \in X, \ f(x,y) \text{ is } \underline{\text{int}} \ \underline{\text{as}} \text{ a } \underline{\text{fn}} \ \underline{\text{of}} \ \underline{y}.$$

Also, $\int_{X\times Y} f \, d\mu = \int_{X\times Y} (f^+ - f^-) \, d\cancel{\mu}\pi \underset{=}{\text{Tonelli}} \int_X \left(\int_Y f^+(x,y) \, d\nu(y) \right) d\mu(x)$

$$- \int_X \left(\int_Y f^-(x,y) \, d\nu(\cancel{x}y) \right) d\mu(x)$$

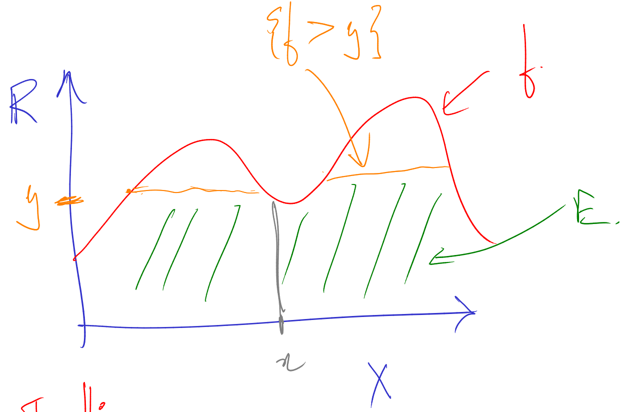
$$= \int_X \left(\int_Y \underbrace{\left(f^+(x,y) - f^-(x,y) \right)}_{f(x,y)} d\nu(y) \right) d\mu(x)$$

$\Rightarrow$ QED Fubini !

Theorem 10.5 (Layer Cake). *If* $f : X \to [0, \infty]$ *is measurable then* $\int_X f \, d\mu = \int_0^\infty \mu(f > t)\, dt.$

(Assume μ is σ-finite)

Pf: Let π = product measure on $\underset{\mu}{X} \times \underset{\lambda}{\mathbb{R}}$.



$E \subset X \times Y = \{(x, y) \mid 0 \leq y \leq f(x)\}.$

$$\pi(E) = \int_{x \in X} \left(\int_{y \in \mathbb{R}} 1_E \, d\lambda(y) \right) d\mu(x) \overset{\text{Tonelli}}{=} \int_{y \in \mathbb{R}} \left(\int_{x \in X} 1_E(x, y) \, d\mu(x) \right) dy$$

$$= \int_{x \in X} f(x) \, d\mu = \int_{y=0}^{\infty} \mu(f > y) \, dy \quad \text{QED.}$$

Proposition 10.6. *If* $(a_{m,n})$ *are such that* $\sum_{m,n=0}^{\infty} \underline{|a_{m,n}|} < \infty$, *then* $\sum_{\underline{m=0}}^{\infty} \sum_{n=0}^{\infty} \underline{a_{m,n}} = \sum_{\underline{n=0}}^{\infty} \sum_{m=0}^{\infty} a_{m,n}$.

Pf: Fubini using the counting measure.

Theorem 10.7 (Minkowski's inequality). *If $f: X \times Y \to \mathbb{R}$ is measurable, then* $p \in [1, \infty]$.

$$\left(\int_X \left|\int_Y f(x,y)\, d\nu(y)\right|^p d\mu(x)\right)^{1/p} \leqslant \int_Y \left(\int_X |f(x,y)|^p\, d\mu(x)\right)^{1/p} d\nu(y)$$

Pf: Intuition:

$$F(x) = \int_{y \in Y} |f(x,y)|\, d\nu(y).$$

Guess

$$\|F\|_{L^p(X)} \leq \int_{y \in Y} \|f(\cdot, y)\|_{L^p(X)}\, d\nu(y)$$

Pf: Let $g \in L^q(X)$, $\frac{1}{p} + \frac{1}{q} = 1$

Compute $\int_X \underline{F(x)}\,|g(x)|\,d\mu(x) = \int_X \int_{y\in Y} |f(x,y)|\,d\nu(y)\;|g(x)|\,d\mu(x)$

$$\overset{\text{Tonelli}}{=} \int_{y\in Y} \left(\int_{x\in X} |f(x,y)|\,|g(x)|\,d\mu(x) \right) d\nu(y)$$

$$\overset{\text{Holder}}{\leq} \int_{y\in Y} \left(\underbrace{\|g\|_{L^q(X)}} \cdot \|f(\cdot,y)\|_{L^p(X)} \right) d\nu(y)$$

$$\leq \|g\|_{L^q(X)} \cdot \overbrace{\int_{y\in Y} \|f(\cdot,y)\|_{L^p(X)}\,d\nu(y)}$$

$\therefore$ By Duality, $\|F\|_{L^p} \leq \int_{y \in Y} \|f(\cdot, y)\|_{L^p(\underline{X})} \, d\nu(y)$

QED

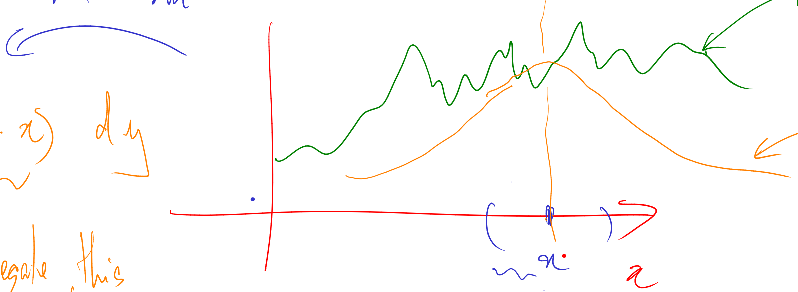
10.2. **Convolutions.**

Definition 10.8. If $f, g \in L^1(\mathbb{R}^d)$ define the *convolution* by $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$.

Remark 10.9. If $f, g \in L^1(\mathbb{R}^d)$, then $f * g < \infty$ almost everywhere.

Smoothed version

$$F(x) = \int_{\mathbb{R}} f(y)\,\varphi(y-x)\,dy$$



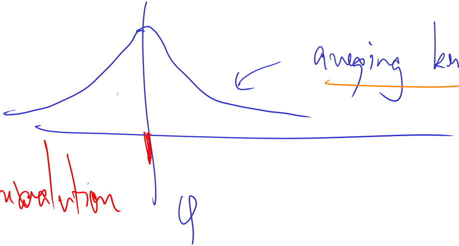
Rough fn f.

φ translated

negate this (helps later)

$$\boxed{F(x) = \int_{\mathbb{R}} f(y)\,\varphi(x-y)\,dy}$$

Convolution



averaging kernel $\left(\varphi \geq 0,\ \int \varphi = 1\right)$

Note: $\int_{x\in\mathbb{R}^d} \left(\int_{y\in\mathbb{R}^d} |f(y)| \; |g(x-y)| \; dy \right) dx$

(Tonelli) $= \int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} |f(y)| \, |g(x-y)| \, dx \quad dy$

$= \int_{y\in\mathbb{R}^d} |f(y)| \; \|g\|_1 \; dy \leq \|f\|_1 \, \|g\|_1$

$\Rightarrow |f*g| < \infty$ a.e. (In fact $\|f*g\|_1 \leq \|f\|_1 \, \|g\|_1$)

Theorem 10.10 (Young). *If* $\boxed{\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}}$, $f\in L^p(\mathbb{R}^d)$, $g\in L^q(\mathbb{R}^d)$ *then* $f*g\in L^r(\mathbb{R}^d)$, *and* $\|f*g\|_{L^r}\leqslant\|f\|_{L^p}\|g\|_{L^q}$.

Remark 10.11. One can show $\|f*g\|_r\leqslant C_{p,q}\|f\|_p\|g\|_q$ for some constant $C_{p,q}<1$. The optimal constant can be found by choosing f,g to be Gaussian's.

Dimension count: $\|f\|_p \sim L^{d/p}$ $\quad f*g \sim L^{d}$

$\|g\|_q \sim L^{d/q}$ $\quad \|f*g\|_r \sim L^{d+d/r}$

Equating dimensions $\frac{\cancel{d}}{p}+\frac{\cancel{d}}{q}=\cancel{d}+\frac{\cancel{d}}{r}$

Pf:

① Use duality. WL. $f,g\geq 0$. Let $h\in L^{r'}$, $h\geq 0$

Let p', q', r' be the Holder conj of p,q,r.

$\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{r}+\frac{1}{r'}=1$

$$② \int_{\mathbb{R}^d} f*g(x)\, h(x)\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\, g(y)\, h(x)\, dy\, dx$$

$$\overset{(*)}{=} \int_{\mathbb{R}^{2d}} \Big(\underbrace{f(x-y)^{\frac{p}{r}}\, g(y)^{\frac{q}{r}}}_{L^{r}} \cdot \underbrace{f(x-y)^{\frac{p}{q'}}\, h(x)^{\frac{r'}{q'}}}_{L^{q'}} \underbrace{g(y)^{\frac{q}{p'}}\, h(x)^{\frac{r'}{p'}}}_{L^{p'}} \Big)\, dx\, dy$$

$$\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r} \Rightarrow 1-\frac{1}{p'}+1-\frac{1}{q'} = 1+\frac{1}{r} \Leftrightarrow \frac{1}{p'}+\frac{1}{q'}+\frac{1}{r} = 1$$

$$\Rightarrow \frac{1}{q'}+\frac{1}{r} = \frac{1}{p} \Rightarrow ① \boxed{\frac{p}{q'}+\frac{p}{r} = 1} \qquad ② \boxed{\frac{q}{p'}+\frac{q}{r} = 1}$$

③ $$\frac{q'}{p'} + \frac{r'}{q'} = 1$$

Holder

$$\circledast \leq \left[\int_{R^{2d}} \left(f(x-y)^{p/r} \, g(y)^{q/r} \right)^{r} dx\,dy \right]^{1/r} \cdot \left[\left(\int \qquad dx\,dy \right)^{q'} \right]^{\frac{1}{q'}}$$

$$\left[\int \left(\quad \right)^{p'} \right]^{1/p'}$$

Tonelli

$$\leq \|f\|_p^{p/r} \, \|g\|_q^{q/r} \cdot \|f\|_p^{\frac{p}{q'}} \, \|h\|_{r'}^{\frac{r'}{q'}} \quad \|g\|_q^{q/p'} \, \|h\|_{r'}^{r'/p'}$$

$$= \|f\|_p \, \|g\|_q \, \|h\|_{r'}$$

Gen Holder : $\sum \frac{1}{p_i} = 1$, $p_i \in [1, \infty]$.

$$\Rightarrow \left| \int \prod f_i \right| \leq \prod \|f_i\|_{p_i} \quad (\text{Holder} + \text{ind}).$$

$$\Rightarrow \left| \int (f * g) \cdot h \right| \leq \|f\|_p \|g\|_q \|h\|_{r'}$$

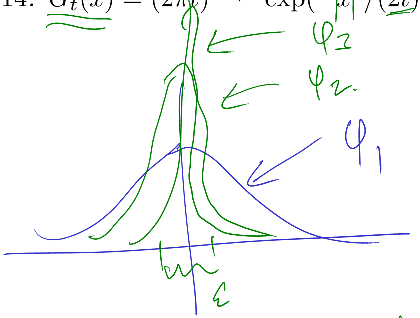
$$\Rightarrow (f*g) \in L^r \ \& \ \|f*g\|_r \leq \|f\|_p \|g\|_q$$

Q.E.D.

Definition 10.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \geqslant 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 10.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$. (A.I. as $\varepsilon \to 0$).

Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-x^2/(2t))$.



$G: \mathbb{R}^d \longrightarrow \mathbb{R}.$

$$G(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$$

$$\forall t > 0, \quad G_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}.$$

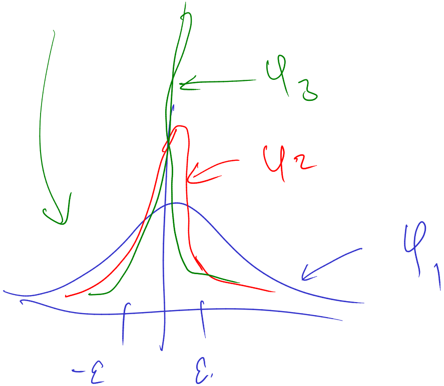
Proposition 10.15. *If* $p \in [1, \infty)$, $f \in L^p$, *and* (φ_n) *is an approximate identity, then* $\underline{\underline{\varphi_n * f} \to f_n}$ *in* L^p.

Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous.

Definition 10.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \geqslant 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 10.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$. ($\varepsilon \to 0$)

Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$, for $x \in \mathbb{R}^d$.



Proposition 10.15. *If $p \in [1, \infty)$, $f \in L^p$, and (φ_n) is an approximate identity, then $\varphi_n * f \to f$ in L^p.*

Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous.

Remk: Useful because (from HW) we know if $\varphi_n \in C_c^\infty \Rightarrow \varphi_n * f \in C^\infty$

Pf of Prop.

$$\varphi_n * f(x) - f(x) = \int_{\mathbb{R}^d} \varphi_n(y)\left[f(x-y) - f(x)\right] dy \qquad \left(\because \int \varphi_n = 1\right)$$

$$\Rightarrow \|\varphi_n * f - f\|_p = \left\| \int_{\mathbb{R}^d} \underline{\hspace{4em}}\, dy \right\|_{L^p} \overset{\text{Minkowski}}{\leq} \int_{y \in \mathbb{R}^d} \varphi(y) \, \|f(\cdot - y) - f(\cdot)\|_{L^p} \, dy$$

Let $\tau_y f(x) = f(x-y)$. $\Rightarrow \|\varphi_n * f - f\|_p \leq \int_{y \in \mathbb{R}^d} \varphi(y) \, \|\tau_y f - f\|_p \, dy$

Pick any $\varepsilon > 0$.

$$\circledast \nearrow \; \|\varphi_n * f - f\|_p \le \int_{|y|<\varepsilon} \varphi_n(y) \underbrace{\|\tau_y f - f\|_p \, dy}_{\text{small when } y \text{ is small!}} + \underbrace{\int_{|y|>\varepsilon} \varphi_n(y) \underline{\underline{\|\tau_y f - f\|_p}}}_{\xrightarrow{n\to\infty} 0}$$

$\forall \delta > 0$, ① $\exists\, \varepsilon > 0 \ni |y| < \varepsilon \Rightarrow \|\tau_y f - f\|_p < \delta$.

② Given ε, $\exists\, n_0$ large $\ni$ $\forall n \ge n_0$, $\displaystyle\int_{|y|>\varepsilon} \varphi_n(y)\, dy < \delta$.

$$① + ② \text{ [illegible] } + \circledast \Rightarrow \|\varphi_n * f - f\|_p \leq \delta \int_{|y|<\varepsilon} \varphi_n(y)\,dy + \delta\left(2\|f\|_p\right)$$

$$\leq \delta\left(1 + 2\|f\|_p\right).$$

QED.

10.3. **Fourier Series.** Let $X = [0,1]$ with the Lebesgue measure. For $n \in \mathbb{Z}$ define $e_n(x) = e^{2\pi i n x}$, and given $f, g \in L^2(X, \mathbb{C})$ define $\langle f, g\rangle = \int_X f\bar{g}\, d\lambda$. This defines an *inner product* on $L^2(X)$, and $\|f\|_{L^2}^2 = \langle f, f\rangle$.

($\bar{g} = g$ Complex conj).

Definition 10.17. If $f \in L^2$, $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f by $\hat{f}(n) = \langle f, e_n\rangle$.

Definition 10.18. For $N \in \mathbb{N}$, let $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n$, be the N-th partial sum of the *Fourier Series* of f.

Question 10.19. *Does $S_N f \to f$? In what sense?*

(Finite dim I.P. space: $\{e_1 \cdots e_N\}$ an $\overline{\text{O.N.}}$ basis $(\langle e_i, e_j\rangle = \delta_{ij})$)

then $\forall v \in V$, , $v = \sum_1^N \langle v, e_i\rangle e_i$

$\hat{f}(n)$ -

$\forall p \in \text{span}\{e_{-N}, \cdots e_N\}$

Lemma 10.20. $\langle e_n, e_m \rangle = \delta_{n,m}$.

Corollary 10.21. *Let* $p \in \text{span}\{e_{-N}, \ldots, e_N\}$. *Then* $\langle f - S_N f, p \rangle = 0$. *Consequently,* $\|f - S_N f\|_2 \leqslant \|f - p\|_2$.

Pf: $\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i n x} e^{-2\pi i m x}\, dx = \int_0^1 e^{2\pi i (n-m) x}\, dx = \delta_{m,n}$.

Pf: Note $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n \Rightarrow \langle S_N f, e_m \rangle = \begin{cases} \hat{f}(m) & |m| \le N. \\ 0 & |m| > N. \end{cases}$

$\Rightarrow \forall |m| \le N, \quad \langle S_N f, e_m \rangle = \hat{f}(m) = \langle f, e_m \rangle.$

$\Rightarrow \langle S_N f - f, e_m \rangle = 0 \quad \forall\ |m| \le N$

linearly $\Rightarrow \langle S_N f - f, p \rangle = 0 \quad \forall p \in \text{Span}\{e_{-N}, \cdots e_N\}.$

Also NTS. $\|f - S_N f\|_2 \le \|f - p\|_2 \quad \forall\, p \in \text{span}\{e_{-N}, \dots, e_N\}$.

Pf: Note $f - S_N f = (f - p) + \underbrace{(p - S_N f)}_{\in \text{span}\{e_{-N}, \dots, e_N\}}$.

$$\Rightarrow \langle f - S_N f, p - S_N f \rangle = 0$$

$$\Rightarrow \|f - S_N f\|_2 \le \|f - p\|_2.$$

QED.

Proposition 10.22. *$S_N f = D_N * f$, where $D_N = \dfrac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(\pi x)}$. The functions D_N are called the* Dirichlet Kernels.

Pf: $S_N f(x) = \sum_{-N}^{N} \hat{f}(n)\, e_n(x) = \sum_{-N}^{N} \int_0^1 f(y)\, e^{-2\pi i n y}\, dy \cdot e^{+2\pi i n x}$

$$= \int_0^1 \Bigg(\underbrace{\sum_{-N}^{N} e^{2\pi i n(x-y)}}_{D_N(x-y) =} \Bigg) f(y)\, dy.$$

(sum & get the formula).

$\Rightarrow S_N f(x) = D_N * f(x)$

Proposition 10.23. *Define the* Cesàro sum *by* $\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_n f$. *Then* $\sigma_N f = F_N * f$, *where* $F_N = \dfrac{1}{N}\left(\dfrac{\sin(N\pi x)}{\sin(\pi x)}\right)^2$.

Remark 10.24. The functions F_N are called the *Fejér Kernels.*

Proposition 10.25. *The Fejér kernels are an approximate identity, but the Dirichlet kernels are not.*

$$\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_n f = \frac{1}{N}\sum_0^{N-1} D_n * f = \underbrace{\left(\frac{1}{N}\sum_0^{N-1} D_n\right)}_{\text{Call this } F_N} * f$$

(Have formulae & can sum & check $F_N =$)

Have formula

can check explicitly.

Corollary 10.26. *If $p \in [1, \infty)$ and $f \in L^p$, then $\sigma_N f \to f$ in L^p.*

Corollary 10.27. *If $f \in L^2$ then $S_N f \to f$ in L^2.*

Remark 10.28. If $f \in L^p$ for $p \neq 2$ we need not have $S_N f \to f$ in L^p.

Pf: Note $\|S_N f - f\|_2 \le \|\sigma_N f - f\|_2$ (only for $p = 2$)

($\because \sigma_N f \in \text{span}\{e_{-N}, \dots e_N\}$)

$\|\sigma_N f - f\|_p \longrightarrow 0 \ \forall p \in [1, \infty) \Rightarrow$ QED.

$X = [0,1]$, $f \in L^1(X)$, $\hat{f}(n) = \langle f, e_n \rangle$.

$$\langle f, g \rangle = \int f \bar{g} \quad , \quad e_n(x) = e^{2\pi i n x}$$

$$S_N f = \sum_{-N}^{N} \hat{f}(n)\, e_n \qquad S_N f = D_N * f$$

(D_N – Dirichlet kernel Not an AI)

$$\sigma_N f = \frac{1}{N} \sum_{0}^{N-1} S_n \qquad \boxed{\underline{\sigma_N f} = F_N * f}$$

(F_N → Fejer Kernel are an AI)

$\Rightarrow \forall\, p \in [1, \infty)$, $\underline{\sigma_N f} \longrightarrow f$ in L^p.

Orthogonality $\Rightarrow$ $S_N f \longrightarrow f$ in L^2 ($p = 2$!!).

Theorem 10.28. *If $p \in [1, \infty)$, $f \in L^p$ then $S_N f \to f$ in L^p.* $(p \neq 1)$

Proof. The proof requires boundedness of the Hilbert transform and is beyond the scope of this course. ☐

Theorem 10.29. *If $f \in L^\infty$ and is Hölder continuous at x with any exponent $\alpha > 0$, then $S_n f(x) \to x$.*

Proof. On homework. ☐

Remark 10.30. If f is simply continuous at x, then certainly $\sigma_n f(x) \to f(x)$, but $S_n f(x)$ need not converge to $f(x)$. In fact, for almost every continuous periodic function function, $S_N f$ diverges on a dense $\boxed{G_\delta}$.

Q: $f \in L^\infty$, Must $S_N f \xrightarrow{L^\infty} f$? (No. $S_N f$ is cts $\forall N$ so if f is not cts $(S_N f) \not\to f$ in L^∞).

Q: $f \in C_{per}([0,1])$ (cts periodic)

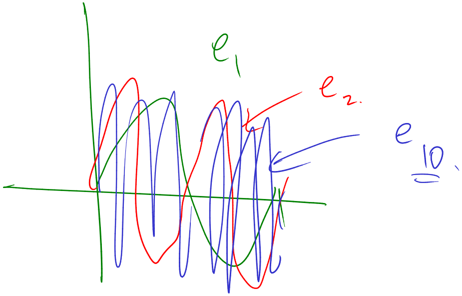
Must $(S_N f) \xrightarrow{L^\infty} f$

The next few results establish a connection between the regularity (differentiability) of a function and decay of its Fourier coefficients.

Theorem 10.31 (Riemann Lebesgue). *Let* μ *be a finite measure and set* $\hat{\mu}(n) = \int_0^1 \overline{e_n}\, d\mu$. *If* $\boxed{\mu \ll \lambda}$, *then* $(\hat{\mu}(n)) \to 0$ *as* $n \to \infty$.

Theorem 10.32 (Parseval's equality). *If* $f \in L^2([0,1])$ *then* $\|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$.

Intuition: More diff a fn is → faster decay of Fourier coefficients.



$$\hat{f}(n) = \langle f, e_n \rangle = \int e^{-2\pi i n x} \underbrace{f(x)\, dx}_{d\mu(x)}$$

Q: μ a finite measure

$$(\hat{\mu}) \in \ell^\infty \Longleftarrow |\hat{\mu}(n)| = \left| \int_0^1 e^{-2\pi i n x}\, d\mu(x) \right| \le \mu([0,1])$$

Pf of Riemann Lebesgue: $\mu \ll \lambda$, By RN, $\exists f \in L^1$ + $d\mu = f\,d\lambda$.

Pick any $\varepsilon > 0$,

① $\exists N$ + $\|f - \sigma_N f\|_{L^1} < \varepsilon$ $\quad$ ($\because (\sigma_N f) \longrightarrow f$ in L^1)

$\rightarrow$② If $g \in L^1$, $|\hat{g}(n)| \le \|g\|_{L^1}$ $\forall n$

$$\Rightarrow \forall n, \quad \left| \underset{\sim}{(f - \sigma_N f)}\hat{\ }(n) \right| \le \|f - \sigma_N f\|_1 < \underline{\underline{\varepsilon}}.$$

③ $\forall n > N$, $(\sigma_N f)\hat{\ }(n) = 0$ $\Rightarrow$ By ②, $|\hat{f}(n)| < \varepsilon$ $\forall n > N$.

QED.

n

Pf of Parseval: $f \in L^2$. NTS $\|f\|_{L^2} = \|\hat{f}\|_{\ell^2}$

More generally, $f, g \in L^2$, then

$$\int_0^1 f\bar{g} = \langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}$$

Pf: $\langle e_n, e_m\rangle = \delta_{m,n}$

$$\Rightarrow \langle S_N f, S_N g\rangle = \sum_{-N}^{N} \hat{f}(n)\overline{\hat{g}(n)}$$

$$S_N f \xrightarrow{L^2} f,\ S_N g \xrightarrow{L^2} g \overset{\text{Holder.}}{\Longrightarrow} \int S_N f\, \overline{S_N g} \longrightarrow \int_0^1 f\bar{g} \Rightarrow \int_0^1 f\bar{g} = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

QED.

Question 10.33. *What are the Fourier coefficients of f'?*

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)\, e_n(x) = \sum_{-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x}$$

Guess $f'(x) = \qquad = \sum \left(\underbrace{2\pi i n\, \hat{f}(n)}\right) e^{2\pi i n x}$

$$\Rightarrow \boxed{(\underline{f'})^{\wedge}(n) \overset{\text{guess}}{=} 2\pi i \underline{n}\, \hat{f}(n).}$$

Definition 10.34. We say g is a *weak derivative* of f if $\langle f, \varphi' \rangle = -\langle g, \varphi \rangle$ for all $\varphi \in C^{\infty}_{per}([0,1])$. ~~($\varphi$ real)~~

Proposition 10.35. *If $f \in L^1$ has a weak derivative $f' \in L^1$, then $(f')^{\wedge}(n) = 2\pi i n \hat{f}(n)$.*

Corollary 10.36. *If $f \in L^2$ has a weak derivative $f' \in L^2$, then* $\boxed{\sum[(1+|n|)|\hat{f}(n)|]^2 < \infty.}$

If f is ~~diff~~ C^1, $\displaystyle\int_0^1 f \overline{\varphi'} \overset{IBP.}{=} -\int_0^1 \varphi \overline{f'}$

$g \leftarrow$ weak derivative.

$$\text{Pf: } (f')^{\wedge}(n) = \langle f', e_n \rangle \overset{IBP}{=} -\langle f, e_n' \rangle$$

$$= -2\pi i n \langle f, e_n \rangle = -2\pi i n \hat{f}(n)$$

QED.

Note: ① $f \in L^1 \Rightarrow R.L. \left(\hat{f}(n)\right) \longrightarrow 0$.

② $f' \in L^1 \Rightarrow \left((f')^{\wedge}(n)\right) \longrightarrow 0 \Rightarrow \left(n \hat{f}(n)\right) \longrightarrow 0$.

Definition 10.37. For $s \geqslant 0$, let $H^s_{per} \overset{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum(1+|n|)^{2s}|\hat{f}(n)|^2$. (Sobolev space of order s).

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

Theorem 10.39 (1D Sobolev Embedding). *If $s > \frac{1}{2}$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when $s = 1/2$.

Remark 10.41. In d dimensions the above is still true if you assume $s > d/2$.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.

Last time: $\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$

Intuition: faster decay of $\hat{f}$ $\approx$ Better regularity of f.

① $f \in L^1 \Rightarrow (\hat{f}(n)) \longrightarrow 0$

② $f \in L^2 \Rightarrow \sum |\hat{f}(n)|^2 < \infty$

③ $f' \in L^2 \Rightarrow \sum |(1+\underline{|n|)\hat{f}}(n)|^2 < \infty \xleftarrow[(s=1)]{}$

Definition 10.37. For $s \geqslant 0$, let $H^s_{per} \overset{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum(1+|n|)^{2s}|\hat{f}(n)|^2$.

H^s = Sobolev space of index s

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

(≈ "s" weak derivatives in L^2)

Theorem 10.39 (1D Sobolev Embedding). *If $s > \frac{1}{2}$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when $s = 1/2$.

Remark 10.41. In d dimensions the above is still true if you assume $\boxed{s > d/2}$.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.

Note: Higher s is $\Rightarrow$ faster decay of $|\hat{f}(n)|$ as $n \to \infty$.

→ Thm + induction $\Rightarrow$ $s > n + \frac{1}{2}$ then $H^s \subseteq C^n_{per}$

(& the incl map is cts)

Pf of 1D Sobolev. $f \in H^s$, $s > \frac{1}{2}$.

Want $\underline{f \in C_{per}}$ ✓ & $\underbrace{\|f\|_\infty \le C \|f\|_{H^s}}_{\text{IOU.}}$ for some const C

① Will show f is cts.

Note that $\underline{f(x)} = \sum \underline{\hat{f}(n)} e^{2\pi i n x}$ in L^2.

(i.e. $\sum \hat{f}(n) e^{2\pi i n x}$ converges in L^2 to f).

<u>Claim</u>: If $f \in H^s$ ($s > 1/2$), then $\sum \hat{f}(n) e^{2\pi i n x}$ conv <u>unif</u>

($\Rightarrow f$ is cts).

Pf of claim: Weierstrass: Enough to show $\sum |\hat{f}(n)| < \infty$

(Note: $\underline{f \in L^2} \Rightarrow \sum |\hat{f}(n)|^2 < \infty \not\Rightarrow \sum |\hat{f}(n)| < \infty$!).

Note $\sum |\hat{f}(n)| = \sum \frac{1}{(1+|n|)^s} (1+|n|)^s |\hat{f}(n)|$

$$\overset{\text{Cauchy Schwartz}}{\le} \left(\sum \frac{1}{(1+|n|)^{2s}} \right)^{1/2} \left(\sum (1+|n|)^{2s} |\hat{f}(n)|^2 \right)^{1/2}.$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{<\infty \text{ when } 2s>1} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{\|f\|_{H^s}.}$$

$< \infty \Rightarrow$ QED.

② WTS $\|f\|_\infty \le C \|f\|_{H^s}$,

Pf: $\|f\|_\infty = \left\| \sum \hat{f}(n) e^{2\pi i n x} \right\|_\infty \le \sum |\hat{f}(n)|$

$$\le \left(\sum \frac{1}{(1+|n|)^{2s}} \right)^{1/2} \cdot \|f\|_{H^s} \Rightarrow \text{QED.}$$

Theorem 10.43 (1D Sobolev embedding)**.** *If* $s > \frac{1}{2} - \frac{1}{2n}$*, then* $H^s_{per} \subseteq L^{2n}$ *and the inclusion map is continuous.* (Next week HW)

Remark 10.44. The above is true for $\boxed{s = \frac{1}{2} - \frac{1}{p}}$, for some $p \in [1, \infty)$ but our proof won't work.

$f \in H^s$, $s > \frac{1}{2} - \frac{1}{2n}$ $\Rightarrow$ $\int_0^1 |f|^{2n} < \infty$

Why is "H^s" stuff useful? (Last Q on the notes HW)

$L^2 \longrightarrow \infty$ dim V.S. $\{\|f\|_{L^2} \leq 1\}$ is not cpt.

$H^1_{per} \subseteq L^2$. <u>Claim:</u> $\{f \in L^2 \mid \|f\|_{H^1_{per}} \leq 1\} \subseteq L^2$ is relatively cpt!

11. Differentiation

11.1. Lebesgue Differentiation.

Theorem 11.1 (Fundamental theorem of Calculus 1). *If* f *is continuous and* $F(x) = \int_0^x f(t)\,dt$*, then* F *is differentiable and* $F' = f$*.*

Theorem 11.2 (Fundamental theorem of Calculus 2). *If* f *is Riemann integrable, and* $F' = f$*, then* $\int_a^b f = F(b) - F(a)$*.*

Our goal is to generalize these to Lebesgue integrable functions.

Theorem 11.3 (Lebesgue Differentiation). *If* $f \in L^1(\mathbb{R}^d)$*, then for almost every* $x \in \mathbb{R}^d$ *we have* $\cancel{\frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f\,d\lambda} = f(x)$*.*

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f\,d\lambda \overset{a.e}{=} f(x)$$

Note: If $d=1$, $f \in L^1(\mathbb{R})$, $F = \int_0^x f$, $\lim_{h \to 0} \frac{F(x+h)-F(x-h)}{2h} = \lim \frac{1}{2h}\int_{x-h}^{x+h} f$

Leb diff $f(x)$ a.e.

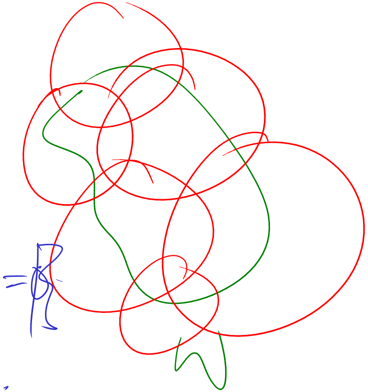
$\forall A \subseteq \mathbb{R}^d$, $|A| = \lambda(A)$

Lemma 11.4 (Vitali Covering Lemma). *Let* $W \subseteq \cup_1^N B(x_i, r_i)$. *There exists* $S \subseteq \{1, \dots, N\}$ *such that:*

→ (1) $\{B(x_i, r_i) \mid i \in S\}$ *are pairwise disjoint.*

(2) $W \subseteq \cup_{i \in S} B(x_i, 3r_i)$ *and hence* $|W| \leqslant 3^d \sum_{i \in S} B(x_i, r_i)$.

Pf: ① Pick n_0 $\ni$ $B(x_{n_0}, r_{n_0})$ has the largest radius.



② Pick n_1 $\ni$ amongst all $\{B(x_i, r_i) \mid B(x_{n_0}, r_{n_0}) \cap B(x_i, r_i) = \emptyset\}$ so that $B(x_{n_1}, r_{n_1})$ has the largest radius.

③ Keep going $r_{n_{k+1}}$ is the largest radius amongst all $B(x_i, r_i)$ that are disj from $B(x_0, r_{n_0}) \cdots B(x_{n_k}, r_{n_k})$

④ Claim this is the desired collection.

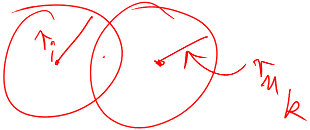
ⓐ clearly $B(x_{n_0}, r_{n_0}), B(x_{n_1}, r_{n_1}) \dots$ are disj by constr.

ⓑ Pick any $B(x_i, r_i)$ which is not amongst

$$\Rightarrow \exists k \ni r_{n_k} \geq r_i \;\&\; B(x_i, r_i) \cap B(x_{n_k}; r_{n_k}) \neq \phi.$$

(by constr)

$$\Rightarrow B(x_{n_k}, 3r_{n_k}) \supseteq B(x_i, r_i)$$



$$\Rightarrow \bigcup_k B(x_{n_k}, 3r_{n_k}) \supseteq \bigcup B(x_i, r_i) \supseteq W$$ QED.

$$\langle S_N f, \cancel{S_N} g \rangle \xrightarrow{\text{Wait}} \langle f, g \rangle.$$

$$|\langle S_N f - f, g \rangle| \overset{\text{Holder}}{\leq} \underbrace{\|S_N f\cancel{\theta} - f\|_{L^2}}_{\substack{\downarrow \\ 0}} \|g\|_{L^2} \longrightarrow 0.$$

Last time: Thm: $f \in L^1(\mathbb{R}^d)$ then $\underline{\tilde{\forall} x \in \mathbb{R}^d}$

$$f(x) = \lim_{\underline{r \to 0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

Last time: Vitali. If $W \subseteq \bigcup_{i=1}^{N} B(x_i, r_i)$, then $\exists$

$B(x_{n_1}, r_{n_1}) \cdots B(x_{n_k}, r_{n_k})$ DISJOINT + $W \subseteq \bigcup_{i=1}^{k} B(x_{n_i}, 3 r_{n_i})$

$$\left(\Rightarrow |W| \leq \underline{3^d} \sum |B(x_{n_i}, r_{n_i})| \right)$$

Definition 11.5 (Maximal function)**.** Let μ be a finite (signed) Borel measure on $\mathbb{R}^d$. Define the *maximal function* of μ by

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}$$

Proposition 11.6. *$M\mu \in L^{1,\infty}$, and $|\{M\mu > \alpha\}| \leqslant \frac{3^d}{\alpha}\|\mu\|$.*

Corollary 11.7. *If $f \in L^1(\mathbb{R}^d)$, then $|\{Mf > \alpha\}| \leqslant \frac{3^d}{\alpha}\|f\|_{L^1}$.*

$f \in L^1(\mathbb{R}^d)$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f|$$

Want

$$f \in L^1 \Rightarrow Mf \in L^1 \ \& \ \boxed{\|Mf\|_{L^1} \leq C\|f\|_{L^1}}$$

(Turns out this is false)

$$f \in L^1 \Rightarrow \forall \alpha \ \left|\{|f| > \alpha\}\right| \leq \frac{\cancel{\|f\|_{L^1}}}{\alpha} C$$

Claim: $\forall p \in (1, \infty]$

$$\exists C_p \ni \|Mf\|_p \leq C_p \|f\|_p$$

In

Pf of Prop: $\left.\begin{array}{l}\mu \to \text{finite signed measure} \\ \alpha > 0\end{array}\right\}$ Want $\underline{\left|\{M_\mu > \alpha\}\right|} \leq \dfrac{C\|\mu\|}{\alpha}$.

① W.L. assume μ is +ve.

② Let $\alpha > 0$. Pick $K \subseteq \{M_\mu > \alpha\}$ cpt. $\left(M_\mu(x) = \sup\limits_{r>0} \dfrac{|\mu(B(x,r))|}{|B(x,r)|} \right)$

$$\Rightarrow \forall x \in K,\ \exists\, r_x \ni \mu(B(x,r_x)) > \alpha\, |\underline{B(x,r_x)}| \ \text{---}\ Ⓧ$$

K cpt $\Rightarrow \exists\, x_1 \cdots x_N \ni K \subseteq \bigcup\limits_{i=1}^{N} B(x_i, r_{x_i})$

Vitali $\Rightarrow \exists\, x_1 \cdots x_M \ni K \subset \bigcup\limits_{i=1}^{M} B(x_i, 3r_{x_i})$ & $\{B(x_i, r_{x_i})\}$ are all disj.

Hence $$|K| \le \left| \bigcup_{1}^{M} B(x_i, 3r_{x_i}) \right| \le 3^d \sum |B(x_i, r_{x_i})|$$

$$\overset{(*)}{\le} \frac{3^d}{\alpha} \sum \mu\big(B(x_i, r_{x_i})\big)$$

$$\overset{(disj)}{=} \frac{3^d}{\alpha} \mu\Big(\bigcup_{1}^{M} B(x_i, r_i) \Big)$$

$$\le \|\mu\| \, 3^d/\alpha$$

QED.

Proposition 11.8. *If $f \in L^1(\mathbb{R}^d)$, then* $\lim\limits_{r\to 0} \frac{1}{|B(x,r)|}\int_{|y-x|<r} |f(y)-f(x)|\,dy = 0$ *almost everywhere.*

Remark 11.9. This immediately implies Theorem 11.3.

Thm (Lebesgue) $\tilde{\forall} x$, $f(x) = \lim\limits_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$.

Pf strategy ① Prove thm for nice fns. (e.g. cts fn).

② $\forall f \in L^1$, write $f = g + h$, $g \to$ nice & $h \to$ small in L^1

③ Obtain a unif bound ~~of~~ for h.

m

Pf: Let $\Omega f(x) = \overline{\lim_{r \to 0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f(y)| \, dy$

① Clearly $f \in C_c$, $\Omega f(x) = 0$ $\forall x$.

② $\forall \varepsilon > 0$, $\exists\, g\ C_c$ & $h \in L^1$ $\ni$ $f = g + h$ & $\|h\|_{L^1} < \varepsilon$.

③ $\Omega f(x) = \Omega h(x) = \overline{\lim_{r \to 0}} \frac{1}{|B(x,r)|} \int_{|B(x,r)|} |h(x) - h(y)| \, dy$

$$\leq |h(x)| + |Mh(x)|$$

$$\Rightarrow \forall \alpha > 0, \quad \left|\{\Omega_f > \alpha\}\right| \leq \left|\{|h\cancel{(x)}| + |Mh\cancel{(x)}| > \alpha\}\right|$$

$$\leq \left|\{|h| > \alpha/2\}\right| + \left|\{Mh > \tfrac{\alpha}{2}\}\right|$$

$$\leq \frac{2\|h\|_{L^1}}{\alpha} + \frac{2 \cdot 3^d}{\alpha}\|h\|_{L^1} \leq \frac{C}{\alpha}\underbrace{\|h\|_{L^1}}_{\varepsilon.}$$

$$\Rightarrow \forall \alpha > 0, \quad \left|\{\Omega_f > \alpha\}\right| \leq \frac{C\varepsilon}{\alpha} \qquad (\varepsilon \text{ is arb})$$

$$\Rightarrow \left|\{|\Omega_f > \alpha\}\right| = 0 \quad \forall \alpha > 0. \quad \Rightarrow \Omega_f = 0 \text{ a.e.}$$

QED

Corollary 11.10. *If $\mu \ll \lambda$ is a finite signed measure, then the Radon-Nikodym derivative is given by* $\dfrac{d\mu}{d\lambda} = \lim_{r\to 0} \dfrac{\mu(B(x,r))}{|B(x,r)|}$.

Remark 11.11. Will use this to prove the change of variables formula.

~~[illegible]~~ RN $\Rightarrow \exists f \in L^1$ s.t. $d\mu = f\, d\lambda$.

L. diff $\Rightarrow$ $f(x) \overset{a.e.}{=} \lim_{r\to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} f$

$$\frac{d\mu}{d\lambda} = f(x)$$

$$= \lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}.$$

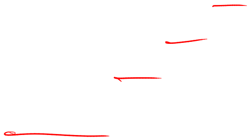
Let's now deal with the second fundamental theorem of calculus:

Question 11.12. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere imply $f' \in L^1$?*

Question 11.13. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere, and $f' \in L^1$ imply $f(x) = \int_0^x f'$?* (No → Cantor fn)

$$\int_a^b f' = f(b) - f(a). \qquad (f' \to \text{R. int})$$

No: Eg $f(x) = \begin{cases} 1/\sqrt{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$



Definition 11.14. We say $f\colon \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^N |x_i - y_i| < \delta \implies \sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.

11.2. Fundamental theorem of calculus.

Question 11.12. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere imply* $f' \in L^1$*?* NO

Question 11.13. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere, and* $f' \in L^1$ *imply* $f(x) = \int_0^x f'$*?* (NO, Cantor fn)

$$f \in L^1_{loc}(\mathbb{R}^d), \qquad \tilde{\forall}\, x \in \mathbb{R}^d, \quad f(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

$$\forall K \subseteq \mathbb{R}^d, \quad f \in L^1(K)$$

$$\int_K |f| < \infty$$

Definition 11.14. We say $f\colon \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^N |x_i - y_i| < \delta \implies \sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.

Note :

Chose $N=1$

(x_i, y_i) are finitely many disj intervals

(Eg : Cantor fn is cts but not a.c.)

Theorem 11.16. *Let $f\colon [a,b] \to \mathbb{R}$ be measurable. Then f is absolutely continuous if and only if f is differentiable almost everywhere, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f'$ ~~almost~~ everywhere.*

Proof of the reverse implication of Theorem 11.16

Assume f diff a.e., $f' \in L^1$, & $f(x) = f(a) + \int_0^x f'$.

NTS f is a.c.

Pf: Let $\varepsilon > 0$. $f' \in L^1 \Rightarrow \exists\, \delta > 0 \ni \mu(E) < \delta \Rightarrow \int_E |f'| < \varepsilon$.

Take $(x_1, y_1) \cdots (x_N, y_N)$ disj $\ni \sum_1^N |x_i - y_i| < \delta$.

$$\Rightarrow \sum_1^N |f(x_i) - f(y_i)| = \sum_1^N \left| \int_{x_i}^{y_i} f' \right| \leq \int_{\bigcup (x_i, y_i)} |f'| < \varepsilon$$

QED.

Lemma 11.17. *If f is absolutely continuous, monotone and injective, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ ~~almost~~ everywhere.*

(W.L. assume f is inc)

Pf: ① Let $\mu(A) = |f(A)|$ $(A \in \mathcal{B})$

Q: $A \in \mathcal{B} \Rightarrow f(A) \in \mathcal{B}$

(Yes) $\Rightarrow$ (f cts inj $\Rightarrow f^{-1}$ cts $\Rightarrow f(A) \in \mathcal{B}$ whenever $A \in \mathcal{B}$)

$\Rightarrow \mu$ is a finite measure.

② Claim: $\mu \ll \lambda$

Pf: Say $A \subseteq [a,b]$, $|A| = 0$, NTS $\mu(A) = 0$

ETS $\forall K \subseteq A$ cpt, $\mu(K) = 0$

Pick any $\varepsilon > 0$. Choose δ as in the def of a.c. of f.

$\exists\, U \supseteq K \ni |U| < \delta$

K cpt $\Rightarrow \exists\, (x_1, y_1) \cdots (x_N, y_N)$ disj $\ni \sum_1^N |x_i - y_i| < \delta$

$$\Rightarrow \mu\left(\bigcup_1^N (x_i, y_i)\right) = \sum_1^N |f(x_i) - f(y_i)| \overset{a.c}{<} \varepsilon \Rightarrow \mu(K) < \varepsilon$$

$$\Rightarrow \mu(A) = \sup_{K \subseteq A} \mu(K) = 0 \Rightarrow \mu \ll \lambda.$$

③ R.N. $\Rightarrow \exists\, g \in L^1$ + $d\mu = g\, d\lambda$.

$$\left.\begin{array}{c}\Rightarrow \mu([a,x]) = f(x) - f(a) \\ \| \\ \int_a^x g(y)\, dy\end{array}\right\} \Rightarrow f(x) = f(a) + \int_a^x g(y)\, dy$$

Lebesgue diff $\Rightarrow f$ is diff a.e. & $f' = g$ a.e.

$$\Rightarrow f(x) = f(a) + \int_a^x f' \quad \text{a.e.}$$

Q.E.D.

Lemma 11.18. *If f is absolutely continuous and monotone, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ almost everywhere.*

Pf: W.L, assume f is inc.

Let $g(x) = f(x) + x$. Clearly g is strictly inc, & a.c.

$\Rightarrow$ FTC holds for g. i.e. $g(x) = g(a) + \int_a^x g'$

$$\Rightarrow f(x) = g(a) + \int_a^x g' - x$$

$\Rightarrow f$ is diff a.e. & $f' = g' - 1$ $\Rightarrow$ QED.

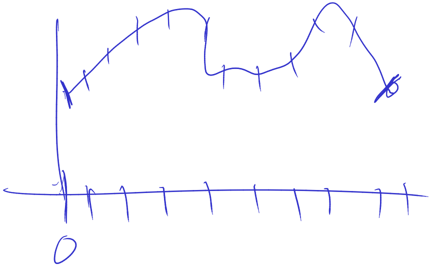
inc

Lemma 11.19. *If f is absolutely continuous then there exist g, h monotone such that $f = g - h$.*

Proof of the forward implication of Theorem 11.16. Follows immediately from the previous lemmas. □

Pf: Claim A.C. ⇒ f has finite variation. (W.L. $[a,b] = [0,1]$.)

$\mathrm{Var}(f) = \sup \sum |f(x_{i+1}) - f(x_i)|$, where $\{x_0, \dots x_N\}$ is a partition of $[0,1]$.



Note: A.C. ⇒ $\mathrm{Var}(f) < \infty$.

Pf: Pick $\varepsilon = 1$. $\exists N$ + $\sum |y_i - x_i| < \frac{1}{N}$

$\Rightarrow \sum |f(x_i) - f(y_i)| \le 1$.

$\boxed{\text{Claim: } \mathrm{Var}(f) \le N}$

Claim: If f has finite var

$\Rightarrow \exists\, g, h$ inc s.t. $f = g - h$.

Pf: Let $F(x)$ = var of f on $[0, x]$

$= \sup \sum |f(x_{i+1}) - f(x_i)|$ over all finite part of $[0, x]$.

Clearly: ① F is increasing

(Requires checking) → ② $F + f$ & $F - f$ are both increasing.

③ f is a.c. $\Rightarrow$ F is a.c. (immediate)

Hence : If f is a.c.

Define $F(x) =$ var of f on $[0, x]$

$$\Rightarrow f = \underbrace{\frac{(F+f)}{2}}_{\text{a.c. \& inc}} - \underbrace{\frac{(F-f)}{2}}_{\text{a.c. \& inc.}}$$

$\Rightarrow$ FTC holds for f.

Q.E.D.

11.3. Change of variables.

Theorem 11.20. *Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi\colon U \to V$ be C^1 and bijective. If $f \in L^1(V)$, then* $\int_V f\,d\lambda = \int_U f \circ \varphi |\det \nabla\varphi|\,d\lambda$.

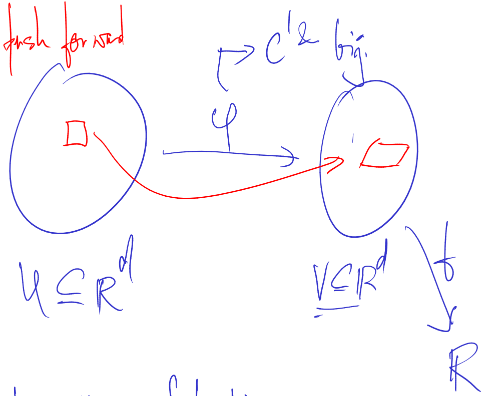
The main idea behind the proof is as follows: Let $\boxed{\mu(A) = \lambda(\varphi(A))}$.

Lemma 11.21. *μ is a Borel measure and* $\int_U f \circ \varphi\,d\mu = \int_V f\,d\lambda$. ← (true from push forward measures)

Lemma 11.22. $\mu \ll \lambda$

Lemma 11.23. $D\mu(x) = |\det \nabla\varphi(x)|$, *where* $D\mu(x) = \lim_{r\to 0} \dfrac{\mu(B(x,r))}{|B(x,r)|}$.

Proof of Theorem 11.20. Follows immediately from the above Lemmas. ☐



$$\int_U f\circ\varphi\,|\det D\varphi|\,d\lambda = \int_V f\,d\lambda$$

Proof of Lemma 11.21

Only NTS μ is a Borel meas. (μ is certainly a meas)

Only NTS $\forall A \in \mathcal{B}(U)$, $\varphi(A) \in \mathcal{B}(V)$.

Note ① $\Sigma = \{A \mid \varphi(A) \in \mathcal{B}(V)\}$ is a σ-alg.

② $\Sigma \supseteq$ all opt sets ($\Rightarrow$ all closed & bdd cells)

$\Rightarrow$ $\Sigma \supseteq \mathcal{B}(U)$ QED.

Proof of Lemma 11.22   NTS $\mu \ll \lambda$. Let $A \subseteq U$, $|A| = 0$, NTS $|\varphi(A)| = 0$

ETS $\forall K \subseteq U$ cpt, $\lambda(K) = 0 \Rightarrow |\varphi(K)| = 0$

Say $|K| = 0$. Pick $\varepsilon > 0$, Find $W \supseteq K$ open $\ni$ $|W| < \varepsilon$

& $\underline{\underline{\overline{W}}} \subseteq U$ & is cpt. ~~convex~~

Note: $\overline{W}$ cpt $\Rightarrow \sup\limits_{x \in \overline{W}} \underline{|\nabla\varphi(x)|} = c < \infty$.

$$\Rightarrow |\varphi(x) - \varphi(y)| \overset{MVT}{=} |D\varphi(\xi)(x-y)| \leq c|x-y| \quad (\forall x,y \in \cancel{\overline{W}}\ \Delta)$$

Δ some convex subset of $\overline{W}$

(↑ ~~only works if $\overline{W}$ is convex~~).

Can ignore $\longrightarrow$ ~~[If $\overline{W}$ is not convex, Cover $\overline{W}$ by N Balls (each c'tly contained in U). Use the MVT in each ball & get $|\varphi(x)-\varphi(y)| \le \underline{N\cdot c}\,|x-y|$.)~~

Pick balls $B(x_i, r_i) \ni K \subseteq \bigcup_1^N B(x_i, r_i)$ & $B(x_i, 3r_i) \subseteq \underline{\underline{\overline{W}}}$

$\left|\bigcup_1^N B(x_i, r_i)\right| < \varepsilon$. $\overset{\text{Vitali}}{\Rightarrow}$ ∃ a disj subset ∋ $K \subseteq \bigcup_1^M B(x_i, \underline{\underline{3}}r_i)$ & $\sum_1^M |B(x_i, r_i)| \le 3^d \delta$

$$\Rightarrow |\varphi(K)| \le \sum_1^{M} |\varphi(B(x_i, 3r_i))| \le \sum_1^{M} c^d\, |B(x_i, 3r_i)|$$

$$\le \varepsilon\, \underline{c}^d \cdot 3^d.$$

QED.

Proof of Lemma 11.23 NTS $D\mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = |\det \nabla \varphi(x)|$

① If $T : \mathbb{R}^d \to \mathbb{R}^d$ is linear know $|T(A)| = |\det T|\,|A|$

② Pick $x_0 \in U$, Case I : $\nabla \varphi(x_0)$ is inv.

Using ① can W.L. assume $\nabla \varphi(x_0) = I$ (Id matrix).

Also, WL, $x_0 = 0$ & $\varphi(0) = 0$

$\Rightarrow |\varphi(x) - x| < \varepsilon |x|$ $\forall x$ small.

$\Rightarrow$ $\forall r$ small, $\varphi(B(0,r)) \subseteq B(0,(1+\varepsilon)r)$

$$\Rightarrow \overline{\lim_{r \to 0}} \frac{|\varphi(B(0,r))|}{|B(0,r)|} \underset{\uparrow}{\leq} (1+\varepsilon)^d$$

Lower bd: Inv fn thm. φ^{-1} is C^1 (near 0).

Inv continuity $\Rightarrow$ $B\left(0, \frac{r}{1+\varepsilon}\right) \subseteq \varphi(B(0,r))$

$$\Rightarrow \underline{\lim}_{r \to 0} \frac{|\varphi(B(0,r))|}{|B(0,r)|} \geq \frac{1}{(1+\varepsilon)^d}.$$

ε is arb $\Rightarrow \lim_{r \to 0} \dfrac{|\varphi(B(0,r))|}{|B(0,r)|} = 1$

QED.

Case 2: $\nabla\varphi(x_0)$ not inv.

(Upper bd pf still works & gives something small).

Please check.

12. **Fourier Transform**

12.1. **Definition and Basic Properties.**

(1) Recall if $f \in L^2_{per}([0,1])$, we set $e_n(x) = e^{2\pi i n x}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx$ and got $f = \sum a_n e_n$ in L^2.

(2) Suppose now $f \in L^2_{per}([-L/2, L/2])$. Can we rescale and send $L \to \infty$?

$$X = \left[-\frac{L}{2}, \frac{L}{2}\right]. \quad e_n(x) = \frac{e^{-2\pi i n x/L}}{\sqrt{L}}. \quad \int_{-L/2}^{L/2} |e_n|^2 = 1$$

$$a_n = \langle f, e_n \rangle = \int_{-L/2}^{L/2} f(x) e^{-2\pi i \left(\frac{n}{L}\right) x} \frac{dx}{\sqrt{L}}. \quad \text{Know } \boxed{f(x) = \sum a_n e_n(x)}$$

Let $\xi = \frac{n}{L}$ send $L \to \infty$ & $n \to \infty$ & hold $\xi = \frac{n}{L}$ constant.

$$\text{Let } \hat{f}(\xi) = \sqrt{L} \cdot a_n = \int_{-L/2}^{L/2} f(x) e^{-2\pi i \xi x}\, dx \xrightarrow{L \to \infty} \overset{\text{F.T.}}{\boxed{\int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi}\, dx}}$$

Also, $f(x) = \sum a_n e_n(x) = \sum \boxed{a_n} \frac{e^{2\pi i \xi x}}{\sqrt{L}} = \underbrace{\sum \hat{f}(\xi) e^{2\pi i x \xi} \cdot \frac{1}{L}}$

$$\xrightarrow{L \to \infty} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi.$$

Define. ~~Guess~~: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx$

Guess: $f(x) = \int_{-\infty}^{\infty} \underline{\hat{f}(\xi) e^{+2\pi i x \xi}} \, d\xi.$

Definition 12.1. If $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, define the *Fourier transform* of f (denoted by $\hat{f}$) by $\hat{f}(\xi) = \int_{\mathbb{R}^d} \overset{f(x)}{\cancel{\hat{f}(\xi)}} e^{-2\pi i \langle x,\xi \rangle}\, dx$

Remark 12.2. More generally, if μ is a finite (signed) Borel measure, then can define $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x,\xi \rangle}\, d\mu(x)$.

Analogous to Fourier series, we will show that $\hat{f}$ is defined even for $f \in L^2$, and prove $\boxed{f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x,\xi \rangle}\, d\xi.}$

$\langle x,\xi \rangle = \sum x_i \xi_i = x \cdot \xi$ (in $\mathbb{R}^d$).

Lemma 12.3 (Linearity). *If* $f, g \in L^1$, $\alpha \in \mathbb{R}$ *then* $(f + \alpha g)^\wedge = \hat{f} + \alpha \hat{g}$.

Lemma 12.4 (Translations). *Let* $\tau_y f(x) = f(x - y)$. *Then* $(\tau_y f)^\wedge(\xi) = e^{-2\pi i \langle y, \xi \rangle} \hat{f}(\xi)$.

Lemma 12.5 (Dilations). *Let* $\delta_\lambda f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. *Then* $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda \xi)$.

$$\Rightarrow (f+\alpha g)^\wedge(\xi) = \int (f+\alpha g)(x)\, e^{-2\pi i \langle x,\xi\rangle} dx = \int f(x)\, e^{-2\pi i\langle x,\xi\rangle} + \alpha \int g(x)\, e^{(\ )}$$

$$= \hat{f}(\xi) + \hat{g}(\xi)$$

Pf: $(\tau_y f)^\wedge(\xi) = \int f(x-y)\, e^{-2\pi i\langle x,\xi\rangle} dx = \underbrace{\left(\int f(x-y)\, e^{-2\pi i\langle x-y,\xi\rangle} dx\right)}\, e^{-2\pi i\langle y,\xi\rangle}$

$$= \hat{f}(\xi)\, e^{-2\pi i\langle y,\xi\rangle}$$

Pf: $(\delta_\lambda f)^\wedge(\xi) = \int \frac{1}{\lambda^d} f\left(\frac{x}{\lambda}\right) e^{-2\pi i \langle \frac{x}{\lambda}, \lambda \xi \rangle} dx = \int f(y)\, e^{-2\pi i \langle y, \lambda \xi \rangle} dy$

$= \hat{f}(\lambda \xi)$ ✓

Lemma 12.6. *If* $\underline{f, g \in L^1}$, *then* $(\underline{f * g})^\wedge = \hat{f}\hat{g}$.

Pf: Note $f, g \in L^1 \to f * g \in L^1$ (Young / ~~Fubi~~ Tonelli)

$$(f*g)^\wedge(\xi) = \int f*g(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx = \int_{x\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} f(y)\, g(x-y)\, e^{-2\pi i \langle x, \xi \rangle}\, dy\, dx$$

$$= \int_{x\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} f(y)\, g(x-y)\, e^{-2\pi i \langle x-y, \xi \rangle}\, e^{-2\pi i \langle y, \xi \rangle}\, dy\, dx$$

$$\overset{\text{Fubini \& Let } z = x-y}{=} \hat{f}(\xi)\, \hat{g}(\xi)$$

QED.

Lemma 12.7. *If* $(1+|x|)f(x) \in L^1(\mathbb{R}^d)$ *then* $\partial_j \hat{f}(\xi) = (-2\pi i x_j f(x))^\wedge(\xi)$.

Lemma 12.8. *If* $f \in C_0^1$, $\partial_j f \in L^1$, *then* $(\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)$.

Decay of $\hat{f}$ $\leftrightsquigarrow$ regularity of f (diff)

diff (regularity) of $\hat{f}$ $\leftrightsquigarrow$ decay of f.

$\lim_{x\to\infty} |f(x)| = 0$

① $(1+|x|)f \in L^1$. WTS $\hat{f}$ is diff &

Fix $j \in \{1,\dots,d\}$,

$$\frac{\hat{f}(\xi+he_j)-\hat{f}(\xi)}{h} = \frac{1}{h}\int\left(f(x)\,e^{-2\pi i\langle x,\xi+he_j\rangle} - f(x)\,e^{-2\pi i\langle x,\xi\rangle}\right)dx$$

$$\circledast \quad = \int f(x)\left(\frac{e^{-2\pi i\langle x,\xi+he_j\rangle} - e^{-2\pi i\langle x,\xi\rangle}}{h}\right)dx.$$

Note $\left| f(x) \cdot \left(\frac{e^{-2\pi i \langle x, \xi + h e_j \rangle} - e^{-2\pi i \langle x, \xi \rangle}}{h} \right) \right| \overset{MVT}{\leq} 2\pi |x_j| \, |f(x)| \in L^1.$

Hence by $\circledast$, $\lim_{h \to 0} \frac{\hat{f}(\xi + x e_j) - \hat{f}(\xi)}{h} \overset{DCT}{=} \int_{\mathbb{R}^d} \underbrace{f(x) \left(-2\pi i x_j\right)}_{} e^{-2\pi i \langle x, \xi \rangle} \, dx$

$$= \left(2\pi i x_j f(x) \right)^{\wedge} (\xi) \quad \checkmark$$

$\square$

Pf: $\underline{f \in C_0^1}$, $\partial_j f \in L^1$ Compute $(\partial_j f)^\wedge(\xi)$.

$$(\partial_j f)^\wedge(\xi) = \int_{\mathbb{R}^d} \cancel{f}(\partial_j f)(x)\, e^{-2\pi i \langle x,\xi\rangle}\, dx$$

$$\overset{\text{By Parts}}{=} + \int_{\mathbb{R}^d} f(x) \left(+ 2\pi i\, \xi_j\, e^{-2\pi i \langle x,\xi\rangle} \right) dx$$

$$= 2\pi i\, \xi_j\, \hat{f}(\xi)$$

Q.E.D.

Theorem 12.9 (Riemann-Lebesgue Lemma)**.** *If* $f \in L^1$*, then* $\hat{f} \in C_0$ *and* $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$*.*

(i.e. $\hat{f}(\xi) \xrightarrow{|\xi| \to \infty} 0$).

Last time: $f \in L^1$, $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$

Goal: $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{+2\pi i \langle x, \xi \rangle}\, d\xi.$

① $(\tau_x f)^\wedge(\xi) = e^{-2\pi i \langle x, \xi \rangle}\, \hat{f}(\xi)$

② $(\partial_j f)^\wedge(\xi) = 2\pi i\, \xi_j\, \hat{f}(\xi)$ & $\overbrace{\partial_j \hat{f}(\xi) = \left(-2\pi i\, x_j f(x)\right)^\wedge(\xi)}$

③ $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda \xi)$ $\left(\delta_\lambda f(x) = \frac{1}{\underline{\lambda^d}} f\left(\frac{x}{\lambda}\right) \right)$

Theorem 12.9 (Riemann-Lebesgue Lemma). *If $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

$$C_0 = \{ f \mid f \text{ is cts } \& \lim_{|x|\to\infty} f(x) = 0 \}$$

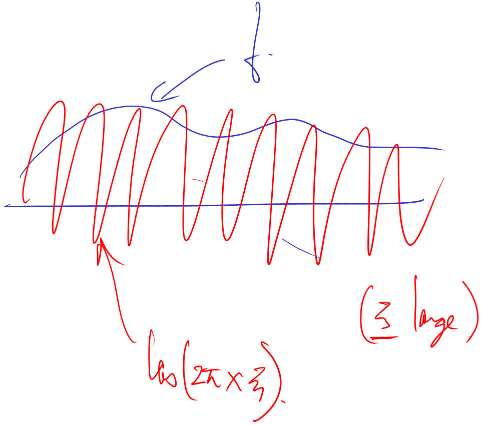
Pf: ① $$|\hat{f}(\xi)| = \left| \int f(x) e^{-2\pi i \langle x,\xi \rangle} dx \right| \le \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_{L^1}$$

② $$\lim_{h\to 0} \hat{f}(\xi+h) = \lim_{h\to 0} \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi+h, x\rangle} dx$$

$$\overset{DC}{=} \int \left(\lim_{h\to 0} f(x) e^{-2\pi i \langle \xi+h, x\rangle} \right) dx = \hat{f}(\xi)$$

$$(\because |f(x) e^{i(\ )}| \le f(x) \in L^1)$$

③ NTS $\hat{f}(\xi) \longrightarrow 0$ as $|\xi| \longrightarrow \infty$.



Pf: $(\tau_x f)^\wedge(\xi) = e^{-2\pi i\langle x,\xi\rangle}\,\hat{f}(\xi)$

$$(f - \tau_x f)^\wedge(\xi) = \Big(1 - \underbrace{e^{2\pi i\langle x,\xi\rangle}}_{-1}\Big)\hat{f}(\xi)$$

$$= 2\hat{f}(\xi).$$

Choose $x = \dfrac{\xi}{2|\xi|^2}$.

$\Rightarrow \quad e^{2\pi i\langle x,\xi\rangle} = e^{i\pi} = -1$

$\Rightarrow$ for $x = \dfrac{\xi}{2|\xi|^2}$, $2\hat{f}(\xi) = (f - \tau_x f)^{\wedge}(\xi)$

$$\Rightarrow 2|\hat{f}(\xi)| \leq \|f - \tau_x f\|_{L^1} \xrightarrow{|x| \to 0} 0$$

$$\Rightarrow \lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0.$$

QED.

12.2. Fourier Inversion.

Theorem 12.10 (Inversion)**.** *If $f, \hat{f} \in L^1$, then* $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle}\, d\xi$. (for a.e. $x \in \mathbb{R}^d$).

Direct proof attempt:

$$\int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}\, d\xi = \int_{\xi \in \mathbb{R}^d} \left(\int_{y \in \mathbb{R}^d} f(y) e^{-2\pi i \langle y, \xi \rangle}\, dy \right) e^{+2\pi i \langle x, \xi \rangle}\, d\xi$$

$$= \int_{\xi \in \mathbb{R}^d} \int_{y \in \mathbb{R}^d} f(y)\, e^{2\pi i \langle x-y, \xi \rangle}\, dy\, d\xi. \quad \text{((Can't Fubini!))}$$

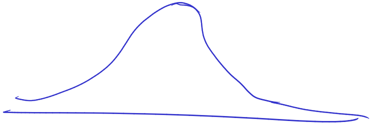
$$\overset{\text{Fubini anyway}}{=} \int_{y \in \mathbb{R}^d} \overbrace{\int_{\xi \in \mathbb{R}^d} f(y)}\, e^{2\pi i \langle x-y, \xi \rangle}\, d\xi\, dy$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\delta_x}$$

$$= \int_{y \in \mathbb{R}^d} f(y)\ \delta(x-y)\ dy = f(x) \qquad \text{“QED”}$$

Correct Pf of inversion ↓

Lemma 12.11. *If* $G(x) = (2\pi)^{-d/2}e^{-|x|^2/2}$, *then* $\underline{\hat{G}(\xi)} = \underline{e^{-|2\pi\xi|^2/2}}$, *and hence* $\boxed{\hat{\hat{G}} = G}$.



Pf: ① Enough to compute $\hat{G}(\xi)$ for $d = 1$

$$\left(\because \hat{G}(\xi) = \hat{G}(\xi_1)\,\hat{G}(\xi_2)\cdots\hat{G}(\xi_d) \quad \because e^{-\sum x_i^2} = \prod e^{-x_i^2}\right)$$

② $G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\qquad G'(x) = -x\,G(x)$

$$\Rightarrow (G')^\wedge(\xi) = -\left(x\,G(x)\right)^\wedge(\xi)$$

$$\Rightarrow 2\pi i\xi\,\hat{G}(\xi) = \frac{1}{2\pi i}\left(-2\pi i x\,G(x)\right)^\wedge(\xi)$$

$$\Rightarrow \quad 2\pi i\, \xi\, \hat{G}(\xi) = \frac{1}{2\pi i} (\hat{G})'(\xi)$$

$$\Rightarrow (\hat{G})'(\xi) = -\underline{\underline{4\pi^2 \xi}}\, \hat{G}(\xi)$$

$$\Rightarrow \hat{G}(\xi) = \underbrace{\hat{G}(0)}_{1}\, e^{-2\pi^2 \xi^2}.$$

$$\boxed{\hat{G}(\xi) = e^{-2\pi^2 \xi^2}.}$$

$$\left(\hat{G}(0) = \int G(x)\, \underbrace{e^{-2\pi i \langle 0, \xi \rangle}}_{1}\, dx = 1\right)$$

$$\left(\text{You check} \Rightarrow \hat{\hat{G}} = G\right).$$

Lemma 12.12. *If $\underline{f, g \in L^1}$ then $\int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}^d} \hat{f}g$.*

Pf: $\int_{\mathbb{R}^d} f(x)\,\hat{g}(x)\,dx = \int_{\mathbb{R}^d} f(x) \int_{y\in\mathbb{R}^d} g(y)\, e^{-2\pi i \langle x,y\rangle}\,dy\,dx$

$$\overset{\text{Fubini}}{=} \int_{y\in\mathbb{R}^d} \underbrace{\int_{x\in\mathbb{R}^d} f(x)\, g(y)\, e^{-2\pi i\langle x,y\rangle}\,dx}\,dy$$

$$\int_{y\in\mathbb{R}^d} g(y)\cdot \hat{f}(y)\,dy$$

QED.

Lemma 12.13. *If* $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and* $\hat{f} \in L^1(\mathbb{R}^d)$, *then* $f(x) = \displaystyle\int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} \, d\xi$.

Pf: ① Prove this for $x = 0$.

i.e. NTS $$\int_{\mathbb{R}^d} \hat{f}(\xi) \, d\xi = f(0)$$

Let $\varphi(x) = G(x)$, $\quad \varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}\right)$

Know $\hat{\varphi}_\varepsilon(x) = \hat{\varphi}(\varepsilon x) = \hat{G}(\varepsilon x)$

$$f(0) = \lim_{\varepsilon \to 0} f * \varphi_\varepsilon(0) = \lim_{\varepsilon \to 0} \int f(x) \, \varphi_\varepsilon(-x) \, dx$$

$$= \lim_{\varepsilon \to 0} \int f(x) \, \varphi_{\varepsilon}(x) \, dx$$

$$= \lim_{\varepsilon \to 0} \int f(x) \, \hat{\hat{\varphi}}_{\varepsilon}(x) \, dx \qquad \left(\because \hat{\hat{G}} = G \right)$$

$$= \lim_{\varepsilon \to 0} \int \hat{f}(x) \, \hat{\varphi}_{\varepsilon}(x) \, dx = \lim_{\varepsilon \to 0} \int \hat{f}(x) \, \hat{\varphi}(\varepsilon x) \, dx$$

$$\overset{DC}{=} \int \hat{f}(x) \, dx \qquad \left(\begin{array}{l} |\hat{\varphi}(\varepsilon x)| \leq 1 \\ \& \ \hat{\varphi}(\varepsilon x) \xrightarrow{\varepsilon \to 0} \hat{\varphi}(0) = 1 \end{array} \right)$$

QED.

Last time:

Inversion: $\underline{f,} \hat{f} \in L^1 \Rightarrow f(x) = \int e^{+2\pi i \langle x, \xi \rangle} \hat{f}(\xi)\, d\xi$

Lemma 12.13. *If* $f \in \boxed{\underline{C}(\mathbb{R}^d)} \cap L^1(\mathbb{R}^d)$ *and* $\hat{f} \in L^1(\mathbb{R}^d)$*, then* $\underbrace{f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle}\, d\xi}$.

Last time: ① $f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi$.

$$\Big(f(0) = \lim f * \varphi_\varepsilon(0) = \lim \int f(x)\, \varphi_\varepsilon(+x)$$

$$= \lim_{\varepsilon \to 0} \int f(x)\, \hat{\hat{\varphi}}_\varepsilon(x)\, dx = \lim_{\varepsilon \to 0} \int \hat{f}(x)\, \underbrace{\hat{\varphi}(\varepsilon x)}_{\downarrow \, 1}\, dy \overset{DCT}{=} \int \hat{f}(x)\, dx \quad ✓$$

$$\boxed{\begin{aligned} \varphi(x) &= G(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2} \\ \varphi_\varepsilon(x) &= \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right) \end{aligned}}$$

$$\boxed{\tau_x f(y) = f(y - x)}$$

② $f(x) = (\tau_{-x} f)(0)$

$$\overset{①}{=} \int_{\mathbb{R}^d} (\tau_{-x} f)^\wedge(\xi)\, d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{+2\pi i \langle x, \xi \rangle}\, d\xi$$

QED.

Proof of Theorem 12.10. Only assume $f \in L^1$, $\hat{f} \in L^1$. NTS $f(x) = \int \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, d\xi$.

$$\left(\varphi_\varepsilon = \frac{1}{\varepsilon^d}\, G\left(\frac{x}{\varepsilon}\right) \right).$$

Note $f(x) \underset{(\text{subseq})}{\overset{a.e}{=}} \lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x)$

$$= \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} (f * \varphi_\varepsilon)^{\wedge}(\xi)\, d\xi \qquad \left(\because f * \varphi_\varepsilon \in C \cap L^1 \ \& \ (f * \varphi_\varepsilon)^{\wedge} = \underset{\text{in } L^1}{\hat{f}} \cdot \underset{\text{in } L^\infty}{\hat{\varphi}_a} \in L^1 \right)$$

$$= \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)\, \underbrace{\hat{\varphi}(\varepsilon \xi)}_{\downarrow}\, d\xi.$$

$$1 \quad \left(\& \ |\hat{\varphi}(\varepsilon \xi)| \le 1 \right)$$

$$\overset{DCT}{=} \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)\, d\xi$$

QED.

Remark 12.14. If $f, \hat{f} \in L^1$, then $\|f - \varphi_\varepsilon * f\|_{L^\infty} \leqslant \|\hat{f} - (\varphi_\varepsilon * f)^\wedge\|_{L^1} \to 0$

Remark 12.15. If $f, \hat{f} \in L^1$ then $\boxed{\hat{\hat{f}}(x) = f(-x).}$

$\Rightarrow$ $f, \hat{f} \in L^1 \Rightarrow \exists\, g \ni f = g$ a.e. & g is cts.

Pf: $f(x) - \varphi_\varepsilon * f(x) = \int \left[\hat{f}(\xi) - (\varphi_\varepsilon * f)^\wedge(\xi)\right] e^{2\pi i \langle x, \xi\rangle}\, d\xi$ (inv).

$$\Rightarrow \|f - \varphi_\varepsilon * f\|_\infty \leq \|\hat{f} - (\varphi_\varepsilon * f)^\wedge\|_{L^1} = \|(1 - \hat{\varphi}(\varepsilon\xi))\hat{f}(\cancel{\varepsilon}\xi)\|_{L^1} \xrightarrow[\varepsilon \to 0]{DCT} 0$$

$\Rightarrow \varphi_\varepsilon * f \xrightarrow{L^\infty} f$ ✓

↳ Let $Rf(x) = f(-x)$

$$(\hat{f})^{\wedge}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{-2\pi i \langle x, \xi \rangle}\, d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i \langle -x, \xi \rangle}\, d\xi \overset{\text{inversion}}{=} f(-x)$$

QED.

12.3. L^2-theory.

Theorem 12.16 (Plancherel). *The Fourier transform extends to a bijective linear isometry on* $L^2(\mathbb{R}^d;\mathbb{C})$.

Note: $f \in L^2$, $\int e^{2\pi i\langle x,\xi\rangle} f(x)\,dx$ may not be defined (in the Lebesgue sense)

($f \in L^2 \not\Rightarrow f \in L^1$ & so $e^{2\pi i\langle x,\xi\rangle} f(x)$ need not $\in L^1(x)$)

Pick $C_c^\infty \subseteq L^2$ (dense). Let $\mathcal{F} f = \hat{f} \quad \forall f \in C_c^\infty$

Claim: $\mathcal{F} : \overset{\mathcal{S}}{\cancel{C_c^\infty}} \hookrightarrow L^2$ is an L^2 isometry! & use this to extend $\mathcal{F}$ to L^2.

Definition 12.17. Define the *Schwartz space*, $\mathcal{S}$, to be the set of all smooth functions such that $\sup_x (1 + |x|^n)|D^\alpha f(x)| < \infty$ for all $n \in \mathbb{N}$ and multi-indexes α.

Remark 12.18. Note $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}$, and so $\mathcal{S}$ is a dense subset of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

$\alpha = (\alpha_1, \dots \alpha_d)$, $\alpha_i \in \mathbb{N}$. $D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$

(multi-index notation for derivative).

$$S = \left\{ f \;\middle|\; \sup_x D^\alpha f(x) (1 + |x|^n) < \infty \;\; \forall \alpha \text{ (multi indexes) } \& \; \forall n \geq 0 \right\}$$

Q: If $f \in C_c^\infty$ does $\hat{f} \in C^\infty$? (Yes → decay of f ⇝ diff of $\hat{f}$)

Q: If $f \in C_c^\infty$ does $\hat{f} \in C_c^\infty$? NO!

$\boxed{\text{Q: If } f \in S \text{ does } \hat{f} \in S \leftarrow \text{Yes}}$

Lemma 12.19. *If* $f, g \in \mathcal{S}$, *then* $\int_{\mathbb{R}^d} f\,\bar{g}\,dx = \int_{\mathbb{R}^d} \hat{f}\,\bar{\hat{g}}\,d\xi$.

(f, g $\mathbb{C}$ valued)

(i.e. $\langle f, g\rangle_{L^2(\mathbb{R}^d, \mathbb{C})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R}^d, \mathbb{C})}$ $\forall f, g \in \mathcal{S}$

Pf: ① $\overline{\hat{g}}(\xi) = \overline{\int g(x)\, e^{-2\pi i\langle x, \xi\rangle}}\, dx = \int \overline{g(-x)}\, e^{+2\pi i\langle -x, \xi\rangle}\, dx$

$= \widehat{\overline{Rg}}(\xi)$ $\quad (Rg(x) = g(-x))$

$$\boxed{\Rightarrow \overline{\hat{g}} = \widehat{\overline{Rg}}}$$

② $\Rightarrow \int \hat{f}\,\overline{\hat{g}}\,d\xi = \int_{\mathbb{R}^d} \hat{f}\,\widehat{\overline{(Rg)}}\,d\xi = \int_{\mathbb{R}^d} \hat{\hat{f}}\,\overline{(Rg)} = \int (Rf)\,\overline{(Rg)}$

Proof of Theorem 12.16

$= \int_{\mathbb{R}^d} fg$ QED.

① $\forall\, f, g \in \mathcal{S}$, know $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$.

② Define $\mathcal{F} f = \hat{f}$ $\forall\, f \in \mathcal{S}$. ($\mathcal{S} \subseteq L^2$ is dense).

③ $\forall\, g \in L^2$, Pick $f_n \in \mathcal{S} \ni (f_n) \xrightarrow{L^2} g$. Define $\mathcal{F} g = \underbrace{\lim \hat{f}_n}_{L^2 \text{ limit.}}$

(Note f_n is Cauchy in $L^2 \cap \mathcal{S} \Rightarrow \hat{f}_n$ is Cauchy in $L^2 \cap \mathcal{S} \Rightarrow$ lim exists).

④ $\langle \mathcal{F} f, \mathcal{F} g \rangle = \lim \langle \hat{f}_n, \hat{g}_n \rangle$ $(f_n, g_n \in \mathcal{S})$

$$= \lim \langle f_n, g_n \rangle = \langle f, g \rangle.$$

$\Rightarrow$ $\mathcal{F}$ is an isometry on L^2.

⑤ Note $\mathcal{F} : L^2 \longrightarrow L^2$ is bijective

(Pf: $\mathcal{F}^2 f = Rf \quad \forall f \in \mathcal{S} \Rightarrow \mathcal{F}^2 f = Rf \quad \forall f \in L^2$

$\Rightarrow \mathcal{F}^4 f = f \quad \forall f \in L^2. \Rightarrow$ bij QED).

Last time: $f \in L^1$, $\hat{f}(\xi) = \int f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$

$$\langle f, g \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \quad \forall\, f, g \in \mathcal{S}$$

$\langle f, g \rangle = \int f \bar{g}$ $\Rightarrow$ $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

$\Rightarrow$ Let $\mathcal{F}f = \hat{f}(\xi)$. $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ is an L^2 isom $\Rightarrow$ $\mathcal{F}$ extends to an isom on L^2

Definition 12.20. Let $s \geqslant 0$ and define the *Sobolev space of index s* by

$$H^s = \{f \in L^2(\mathbb{R}^d) \mid \|f\|_{H^s} < \infty\}, \qquad \text{where} \qquad \|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}.$$

Remark 12.21. A function $f \in H^1$ if and only if f and all first order *weak derivatives* are in L^2.

Remark 12.22. For $s < 0$, one needs to define H^s as the completion of $\mathcal{S}$ under the H^s norm.

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$$

Proposition 12.23. *Let $s \in (0,1)$. Then $f \in H^s$ if and only if* $\int_0^\infty \left(\frac{\|f-\tau_{hv}f\|_{L^2}}{|h|^s}\right)^2 \frac{dh}{h} < \infty$ *for all $v \in \mathbb{R}^d$.*

Remark 12.24. For $s=1$, we instead need $\sup_{h>0} \frac{1}{h}\|f-\tau_{hv}f\|_{L^2} < \infty$.

Remark 12.25. If $s \in (0,1]$, then there exists $C = C(s)$ such that $\|f-\tau_h f\|_{L^2} \leqslant C|h|^s\|f\|_{L^2}$ for all $f \in H^s$, $h \in \mathbb{R}^d$.

$s \in (0,1)$. $\quad f \in H^s(\mathbb{R}^d) \iff \int_0^\infty \left(\frac{\|f-\tau_h f\|_{L^2}}{|h|^s}\right)^2 \frac{dh}{h^d}$

Note: $C^\alpha = \left\{ f \;\middle|\; \sup \frac{|f(x)-f(y)|}{|x-y|^\alpha} + |f(x)| < \infty \right\}$

$(\alpha \in (0,1))$

$$C^\alpha = \{ f \in L^\infty \mid \|f-\tau_x f\|_{L^\infty} \leq C|x|^\alpha \ \forall x \in \mathbb{R}^d \}$$

$(s<1)$ $\quad H^s \ne$ Prop $\Rightarrow \forall f \in H^s, \ \|f-\tau_h f\|_{L^2} \leq o(|h|^s)$

Prop: Say $f : [0, \infty) \to \mathbb{R}$. $$\int_{x \in \mathbb{R}^d} f(|x|) \, dx = c_d \int_{r=0}^{\infty} f(r) \, \underline{r^{d-1}} \, dr$$

$$c_d = \text{`surface area' of } S^{d-1} \hookrightarrow \mathbb{R}^d.$$

$$\left(S^{d-1} = \{ x \in \mathbb{R}^d \mid |x| = 1 \} \right)$$

Pf: ① Say $f \in H^s(\mathbb{R}^d)$ $\quad s \in (0, 1)$

WTS $$\int_{\mathbb{R}^d} \left(\frac{\|f - \tau_h f\|_2^2}{|h|^{2s}} \right) \frac{dh}{|h|^d} < \infty$$

Note $$\int_{\mathbb{R}^d} \frac{\|f-\tau_h f\|_{L^2}^2}{|h|^{2s}} \frac{dh}{|h|^d} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|1-e^{-2\pi i\langle h,\xi\rangle}|^2}{|h|^{2s}} |\hat{f}(\xi)|^2 \, d\xi \, \frac{dh}{|h|^d}.$$

(Choose δ suitably)

$$\leq \int_{\xi\in\mathbb{R}^d} |\hat{f}(\xi)|^2 \left(\underbrace{C\int_{|h|<\delta} \frac{|h|^2|\xi|^2}{|h|^{2s}} \frac{dh}{|h|^d}}_{①} + \underbrace{\int_{|h|\geq\delta} 2 \, \frac{dh}{|h|^{d+2s}}}_{②} \right) \quad \circledast$$

A better way to do this (courtsey Ethan) is to change variables and replace h with $h/|\xi|$

① : $$\int_{|h|<\delta} |\xi|^2 \frac{dh}{|h|^{d+2s-2}} = c_d \int_{r=0}^{\delta} |\xi|^2 \frac{r^{d-1}\,dr}{r^{d+2s-2}} = c_d |\xi|^2 \int_0^{\delta} \frac{dr}{r^{2s-1}}$$

while $2s-1 < 1 \Rightarrow c_d |\xi|^2 \int_0^\delta \frac{dr}{r^{2s-1}} = C |\xi|^2 \left[r^{2-2s} \right]_0^\delta$

$$= C \frac{|\xi|^2}{2s-2}$$

δ

② $\int_{|h|>\delta} \frac{dh}{|h|^{d+2s}} = \int_{r=\delta}^{\infty} \frac{dr}{r^{2s+1}} = \int_{r=\delta}^{\infty} r^{-1-2s} dr = \left[\frac{r^{-2s}}{(-2s)} \right]_\delta^\infty$

$$= \frac{1}{\delta^{2s}}$$

From (*). $$\int_{\mathbb{R}^d} \frac{\|\tau_h f - f\|_{2}^2}{|h|^{2s}} \frac{dh}{|h|^d} \leqslant C \int_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)|^2 \left(\frac{|\xi|^2}{\delta^{2s-2}} + \frac{1}{\delta^{2s}} \right) d\xi.$$

Choose $\delta \ni |\xi|^2 = \delta^{-2}.$ $\iff$ $\delta = \frac{1}{|\xi|}$

$$\leqslant C \int_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)|^2 \left(2 |\xi|^{2s} \right) \leqslant \|f\|_{H^s}^2 < \infty$$

QED.

Theorem 12.26 (Sobolev embedding). *If* $\underline{s} > \underline{d/2}$ *then* $\underline{H^s(\mathbb{R}^d)} \subseteq \underline{C_b(\mathbb{R}^d)}$, *and the inclusion map is continuous.*

$C_b(\mathbb{R}^d) = L^\infty \cap C(\mathbb{R}^d)$.

Pf: Obs 1: If Inversion holds & $\hat{f} \in L^1 \Rightarrow f$ is cts. (DCT).

$$f(x) = \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)$$

Obs 2: $f \in H^s$ & $s > d/2 \Rightarrow \hat{f} \in L^1$

Pf: $\int |\hat{f}(\xi)| = \int (1+|\xi|^2)^{s/2} |\hat{f}(\xi)| \cdot \frac{1}{(1+|\xi|^2)^{s/2}} \, d\xi$.

$$\overset{\text{Cauchy Schwarz}}{\le} \underbrace{\left\| (1+|\xi|^2)^{s/2} \hat{f}(\xi) \right\|_{L^2}}_{\|f\|_{H^s}} \cdot \underbrace{\left(\int \frac{1}{(1+|\xi|^2)^s} \, d\xi \right)^{1/2}}_{\left(c_d \int_{r=0}^{\infty} \frac{r^{d-1} \, dr}{\underbrace{(1+r^2)}_{}{}^{s}} . \right)^{1/2}}$$

Note $\dfrac{r^{d-1}}{(1+r^2)^s} \approx \dfrac{1}{r^{2s-d+1}}$ (r large)

$$s > \frac{d}{2} \iff \underbrace{2s-d+1 > +1} \Rightarrow \int_{r=0}^{\infty} \frac{dr \; r^{d-1}}{(1+r^2)^s} < \infty$$

$$\Rightarrow \int |\hat{f}(\xi)| < \infty \qquad \Rightarrow \text{done}$$

QED

$$\boxed{\mu \perp \lambda .} \qquad \lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = 0 \quad \lambda \text{ a.e.}$$

Theorem 12.26 (Sobolev embedding). *If $s > d/2$ then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous.*

Recall: $\|f\|_{H^s} = \int |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi$, $H_s = \{ f \in L^2 \mid \|\hat{f}\|_{H^s} < \infty \}$

$(s \geq 0)$.

Last time: $f \in S$.

$$\|f\|_{L^\infty} \underset{\uparrow \text{ inversion}}{\leq} \int_{\mathbb{R}^d} |\hat{f}| = \underbrace{\int (1+|\xi|^2)^{s/2} |\hat{f}(\xi)|} \cdot \frac{1}{(1+|\xi|^2)^{s/2}} \, d\xi .$$

$$\overset{C.S.}{\leq} \|f\|_{H^s} \underbrace{\left(\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^s} \, d\xi \right)^{1/2}} .$$

(Note $\int_{|x|>1} \frac{1}{|x|^{d+\varepsilon}} \, dx < \infty \quad \forall \varepsilon > 0 \qquad (x \in \mathbb{R}^d)$)

$$\boxed{\Rightarrow \|f\|_{L^\infty} \le \underline{C} \, \|f\|_{H^s}} \longleftarrow ⊛ \qquad \underline{\forall f \in \mathcal{S}} \quad \left(\text{provided } s > \frac{d}{2}\right).$$

(actually true $\forall f$ s.t. $f, \hat{f} \in L^1$)

Now given $f \in H^s$, Choose a smooth Approx Id, φ_n.

$\rightarrow$ ① $\varphi_n * f \longrightarrow f$ in $\underline{H^s}$ $\quad$ ($\because$ $\hat{\varphi}_n(\xi) \le 1$

& $\hat{\varphi}_n(\xi) \xrightarrow[n\to\infty]{} 1$

$(\varphi_n * f)\hat{}\,(\xi) = \hat{\varphi}_n(\xi) \cdot \hat{f}(\xi)$ & DCT)

ERROR: $\varphi_n * f$ need not $\in L^1$. Instead choose $g_n \in \mathcal{S}$ s.t. $g_n \longrightarrow f$ in H^s & replace $(\varphi_n * f)$ with g_n in the rest of this proof.

② $\Rightarrow$ $\varphi_n * f$ is Cauchy in H^s

③ $\Rightarrow$ $\varphi_n * f$ is Cauchy in L^∞ (∵ by ⊛)

④ $\Rightarrow$ $\underline{\varphi_n * f} \xrightarrow{L^\infty} f$ ($\Rightarrow \exists\, g$ cts $\ni$ $g = f$ a.e.)

($\Rightarrow H^s \subseteq L^\infty$ for $s > d/2$.)

⑤ Incl map is cts ∵ ⊛ holds $\forall f \in S \cap H^s \underset{\text{dense}}{\subseteq} H^s$

QED.

Corollary 12.27. *If $s > \underline{n + d/2}$, then $\underline{H^s(\mathbb{R}^d)} \subseteq \underline{C_b^n(\mathbb{R}^d)}$ and the inclusion map is continuous.*

$C_b^n = \{ f \in C^n(\mathbb{R}^d) \mid f$ & all nth ord derivatives are bdd $\}$

Pf: ① Say $n = 1$.

$\boxed{f \text{ nice.}}$

$$\|\partial_i f\|_{L^\infty} \leq C \|\partial_i f\|_{H^{s-1}} \quad \left(\text{Embedding} \because s - 1 > \tfrac{d}{2}\right)$$

$$\leq C \|f\|_{H^s}$$

② $\partial_i(\varphi_n * f)$ Cauchy in H^s ⇒ Cauchy in L^∞ ⇒ $\partial_i(\varphi_n * f) \xrightarrow{L^\infty} g_i$

Integrate in x_i & get $g_i = \partial_i f$. ⇒ $f \in C^1$

QED.

Proposition 12.28 (Elliptic regularity)**.** *Say* $f \in \mathcal{S}(\mathbb{R}^d)$*,* $u \in H^2(\mathbb{R}^d)$ *is such that* $\lim_{|x|\to\infty} |x|^d |\nabla u(x)| = 0$ *and* $\boxed{-\Delta u = f}$*, then* $u \in \mathcal{S}$*.*

C_c^∞

Note: only need $u \in C^2$ to make sense of $-\Delta u = f$.

$$\Delta u = \sum_{i=1}^{d} \partial_i^2 u.$$

Pf: $-\Delta u = f \Rightarrow -(\Delta u)^\wedge = \hat{f}$

$$\Rightarrow -\left(\sum \partial_j^2 u\right)^\wedge(\xi) = \hat{f}(\xi) \Rightarrow +4\pi^2 |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{4\pi^2 |\xi|^2}.\Big\}$$

If $\Rightarrow \hat{u}(\xi)$ decays 2 degrees faster than $\hat{f}(\xi)$ as $|\xi| \longrightarrow \infty$

$$\Rightarrow \text{ If } \int_{\underline{\underline{|\xi|>1}}} (1+|\xi|^2)^{2s} \, |\hat{f}(\xi)|^2 < \infty$$

$$\Rightarrow \int_{\underline{\underline{|\xi|>1}}} (1+|\xi|^2)^{2(s+2)} \, |\hat{u}(\xi)|^2 \, d\xi < \infty .$$

How $_{u}$ about $\displaystyle\int_{|\xi|<1}$: Recall $\hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{4\pi^2 \, |\xi|^2}$.

$$\left.\begin{array}{ll} \text{Obs 1:} & \hat{f}(0) = 0 \\ \text{Obs 2:} & \partial_j \hat{f}(0) = 0 \\ \text{Obs 3:} & \hat{f} \in C^2. \end{array}\right\} \Longrightarrow \frac{\hat{f}(\xi)}{|\xi|^2} \text{ remains bdd as } \xi \longrightarrow 0.$$

Pf of 1: Know $-\Delta u = f$ $\Rightarrow \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} (-\Delta u) = \lim_{R\to\infty} -\int_{|x|<R} \Delta u$

$$= \lim_{R\to\infty} -\int_{|x|<R} \nabla\cdot \nabla u.$$

div thm $= \lim_{R\to\infty} \int_{\partial B(0,R)} \nabla u \cdot \hat{n} \, d\sigma \longrightarrow 0$ by assumption $\Rightarrow$ Obs 1.

Obs 2: $-\Delta u = f \Rightarrow -x_j \Delta u = x_j f.$

Int & use div thm get $\int_{\mathbb{R}^d} \underbrace{x_j f(x)} \, dx = 0$

$\Rightarrow \partial_j \hat{f}(0) = 0$ $\Rightarrow$ Obs 2.

QED.

not on final

Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

$$H^*_{\alpha,\delta}(A) = \inf\Big\{\sum_1^\infty \rho_\alpha(E_i) \Bigm| \operatorname{diam}(E_i) < \delta\,, \text{ and } A \subset \bigcup_1^\infty E_j\Big\}\,, \quad \text{where} \quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1 + \frac{\alpha}{2})}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta\to 0} H^*_\alpha$.

Proposition A.3 (From homework 2)**.** *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*

Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

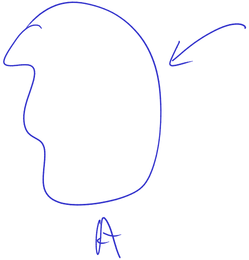
$$H^*_{\alpha,\delta}(A) = \inf\Big\{\sum_1^\infty \rho_\alpha(E_i) \;\Big|\; \operatorname{diam}(E_i) < \delta\,,\ \text{and } A \subset \bigcup_1^\infty E_j\Big\}, \quad \text{where} \quad \rho_\alpha(A) = \underbrace{\frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}}_{C_\alpha}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta\to 0} H^*_\alpha$.

Proposition A.3 (From homework 2). *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*



$$\lambda(A) = \lambda^*(A) = \inf\Big\{ \sum |I_k| \;\Big|\; \bigcup I_k \supseteq A,\ I_k \text{ cells} \Big\}.$$

$$H_d(A) = \lim_{\delta\to 0}\big(\text{sets of diam} \leq \delta.\big)$$

Lemma A.5 (Infinite version of Vitali's Covering Lemma)**.** *Let* $W \subseteq \cup_{\alpha \in A} B(x_\alpha, r_\alpha)$, *with* $\boxed{\sup r_\alpha < \infty.}$ *There exists a countable set* $\mathcal{I} \subseteq A$ *such that:*

(1) $\{B(x_i, r_i) \mid i \in \mathcal{I}\}$ *are pairwise disjoint.* (contable)
(2) $W \subseteq \cup_{i \in \mathcal{I}} B(x_i, 5r_i)$ *and hence* $|W| \leqslant 5^d \sum_{i \in S} B(x_i, r_i)$.

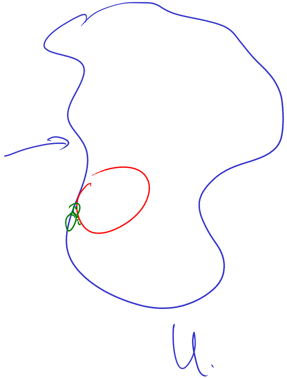
(Similar to finite Vitali Pf).

Lemma A.6. *Let* $U \subseteq \mathbb{R}^d$ *be open and* $\delta > 0$. *There exists countably many* $x_i \in U$, $r_i \in (0, \delta)$ *such that* $\overline{B(x_i, r_i)} \subseteq U$, *are pairwise disjoint, and* $|U - \cup\overline{B(x_i, r_i)}| = 0$.

$$U = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

$$Q: \quad U = \bigcup_{i=1}^{\infty} \underbrace{B(x_i, r_i)}_{\text{disj?}}$$

Claim: Yes except for a null set!



Pf: <u>Lemma</u>: Fix $\delta > 0$.

$$\exists\ \alpha < 1 \quad \& \quad B(x_1, r_1) \cdots B(x_n, r_n) \subseteq U \ \ni\ r_i \in (0, \delta).$$

$$\left| U - \bigcup_{i}^{n} \overline{B(x_i, r_i)} \right| < \alpha\, |U| \qquad \forall\, U \subseteq \mathbb{R}^d, \text{ open } |U| < \infty.$$

Pf of Lemma A.6: ① Say $|U| < \infty$.

② Set $U_0 = U$. $\quad U_{n+1} = U_n - \bigcup_{i}^{m_n} \overline{B(x_{n,i}, r_{n,i})}$ ((from Lemma).

$$|U_{n+1}| < \alpha |U_n| \Rightarrow |U_n| < \alpha^n |U_0| \xrightarrow[n \to \infty]{} 0.$$

③ Let $N = \bigcap_{i}^{\infty} U_n$. Then $U = N \cup \left(\bigcup_{n,i} \overline{B(x_{n,i}, r_{n,i})} \right)$.

Null. pairwise disj (by constr)

Pf of Lemma: ① $U = \bigcup_{x \in U} B(x, r_x)$ & choose $r_x < \delta$

② Vitali $\Rightarrow \exists$ $B(x_i, r_i)$ ctble + pairwise disj $\ni$

$$U \subseteq \bigcup B(x_i, 5r_i) \Rightarrow |U| \leq \underbrace{\sum |B(x_i, r_i)|}_{} \cdot 5^d$$

$$\Rightarrow \frac{1}{5^d}|U| \leq \left| \bigcup_{i}^{\infty} B(x_i, r_i) \right|$$

For N large, $\bigcup_{1}^{N} B(x_i, r_i) \geq \frac{1}{6^d} |U|$.

Set $V = U - \bigcup_{1}^{N} \overline{B(x_i, r_i)}$.

Note $|V| \leq \left(1 - \frac{1}{6^d}\right) |U|$. QED.

Lemma A.7. $H_d \leqslant \lambda$.

Pf: Fix $\delta > 0$, $U \subseteq \mathbb{R}^d$ open.

$$H_{d,\delta}(U) = H_{d,\delta}\Big(\underbrace{N}_{\lambda(N)=0} \cup \underbrace{\Big(\bigcup_{i=1}^{\infty} B(x_i, r_i)\Big)}_{\text{disj } \& \ r_i \leq \delta.}\Big) \quad \text{(by lemma)}$$

$$\leq H_{d,\delta}(N) + \sum_{i=1}^{\infty} H_{d,\delta}\big(\overline{B(x_i, r_i)}\big)$$

$$\leq H_{d,\delta}(N) + \sum_{i=1}^{\infty} \big|\lambda\big(\overline{B(x_i, r_i)}\big)\big| = \boxed{H_{d,\delta}(N)} + |U|.$$

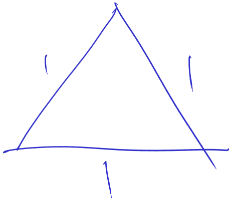
$$\boxed{H_{d,\delta}(N)} = 0 \quad \text{(You check).}$$

$$\Rightarrow \lim_{\delta \to 0} : H_d(U) \leq |U|.$$

Theorem A.8 (Isodiametric inequality)**.** $|A| \leqslant |B(0,1/2)|\,\mathrm{diam}(A)^d = |B(0,\mathrm{diam}(A)/2)|$.

Remark A.9. Note A need not be contained in a ball of radius $\mathrm{diam}(A)/2$.

Eg $\quad A =$



$\mathrm{diam}(A) = 1$

But $A \not\subseteq B(x, \frac{1}{2}) \quad \forall x \in \mathbb{R}^2$.

Proof of Theorem A.4. ($H_d = \lambda$).

↳ Already know $H_d \leq \lambda$.

Reverse: Cover U by sets of diam $\leq \delta$.

$$U \subseteq \bigcup_{1}^{\infty} E_k \Rightarrow |U| \leq \sum |E_k| \underset{\boxed{\text{iso diam}}}{\leq} \sum \ell_d(E_k)$$

inf over all covers $\Rightarrow |U| \leq H_{d,\delta}(U)$ send $\delta \to 0$

QED.

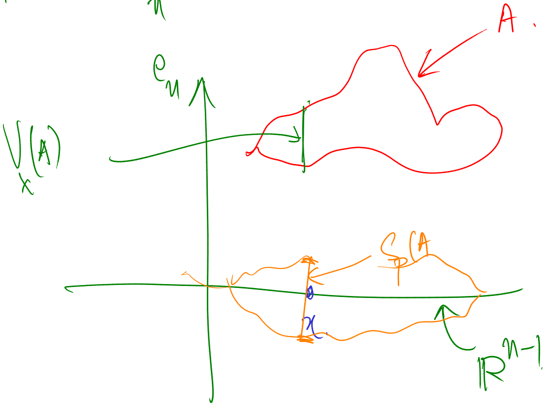
Proposition A.10 (Steiner Symmetrization). *Let $P \subseteq \mathbb{R}^d$ be a hyperplane with unit normal $\hat{n}$. Let $A \in \mathcal{L}(\mathbb{R}^d)$. There exists $S_P(A) \in \mathcal{L}(\mathbb{R}^d)$ such that:*

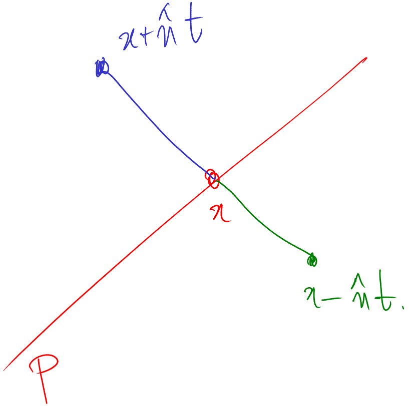
(1) *$S_P(A)$ is symmetric about P (i.e. for any $x \in P$, $t \in \mathbb{R}$, we have $x + t\hat{n} \in \overset{S_P(A)}{\cancel{A}} \iff x - t\hat{n} \in \overset{S_P(A)}{\cancel{A}}$).*

(2) *$\operatorname{diam}(S_P(A)) \leqslant \operatorname{diam}(A)$.*

(3) *$|S_P(A)| = |A|$.*

Pf (i) WL. $P = e_n^{\perp}$.





② fix $x \in \mathbb{R}^{n-1}$. $V_x(A) = \{ y \mid (x,y) \in A \}$.

Set $S_p(A) = \{ (x,y) \mid x \in \mathbb{R}^{n-1} ,\ |y| \leq \frac{1}{2} |V_x(A)| \}$.

③ Clearly $S_p(A)$ is symm abt $e_n^{\perp}$.

④ $|S_p(A)| = |A|$ (Fubini/Tonelli)

⑤ $\mathrm{diam}(S_p(A)) \leq \mathrm{diam}(A)$ (pf by picture) (You check).

Proof of Theorem A.8 (NTS. $|A| \leq \left| B\left(0, \frac{\mathrm{diam}(A)}{2}\right) \right|$.

Pf: Let $B = S_{e_1^\perp}\left(S_{e_2^\perp}\left(\left(\cdots S_{e_n^\perp}(A)\right)\right)\right)$

B is symm about all coordinate axis.

$\Rightarrow x \in B \Leftrightarrow -x \in B.$ $\left(\Rightarrow \mathrm{diam}(B) \geq 2|x| \right)$

$\Rightarrow B \subseteq B\left(0, \frac{\mathrm{diam}(B)}{2}\right) \Rightarrow |B| \leq \left| B\left(0, \frac{\mathrm{diam}(B)}{2}\right) \right|$

$\Rightarrow |A| \leq B\left(0, \frac{\mathrm{diam}(A)}{2}\right)$ QED.