$$\mathcal{L}f(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$g : \mathbb{R} \to \mathbb{R}$.

$$g(t) = \mathbb{1}_{\{t \geq 0\}} f(t)$$

$$\hat{g}(-is) = \int_{\mathbb{R}} \underbrace{g(x)\, e^{-2\pi i \langle -is, x \rangle}\, dx}_{\mathcal{L}f(2\pi s)}$$

Q: $\lim_{s \to \infty} sF(s)$. $F(s) = \int_0^\infty e^{-st} f(t)\, dt$, $f \in$ w.c.

$$\int_0^\infty s e^{-st}\, dt = 1 \qquad sF(s) = \int_0^\infty \underbrace{s e^{-st}} f(t)\, dt$$

$$\int_\varepsilon^\infty s e^{-st}\, dt = -e^{-st}\Big]_\varepsilon^\infty = e^{-\varepsilon s} - 0 \xrightarrow{s \to \infty} 0$$

$\Rightarrow s e^{-st} \longrightarrow$ is A.I.

$$\left.\begin{aligned} &\int_0^\infty s e^{-st}\big(f(t)-f(0)\big)\,dt + f(0) \\ &\searrow \int_0^\varepsilon + \int_\varepsilon^\infty \end{aligned}\right\} \longrightarrow f(0)$$

$f \in \underline{C(X)}$ $\boxed{\mu \text{ a finite measure.}}$ $|\mu|(X)$

$$\Lambda(f) = \int_X f\,d\mu \qquad |\Lambda f| \le \|f\|_\infty \|\mu\|_{TV}$$

Λ is the functional

$f \geq 0 \Rightarrow \Lambda f \geq 0$

Hölder $\|f\|_p = \|g\|_q = 1$

$\frac{1}{p} + \frac{1}{q} = 1$

$\int |fg| \leq \int \left(\frac{|f|^p}{p} + \frac{|g|^q}{q} \right) = 1$

Young.

$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$

(equality holds only if $x^p = y^q$)

$\Rightarrow$ Equality holds in Hölder $\Leftrightarrow f = \lambda g$

Uncertainty Principle:

$$\underbrace{\|x f(x)\|_{L^2}}_{\text{"variance in position"}} \ \underbrace{\|\xi \hat{f}(\xi)\|_{L^2}}_{\text{"variance in momentum"}} \geq c \, \|f\|_{L^2} \, \|\hat{f}\|_{L^2}$$

$$\left(\int f \bar{g} = \int \hat{f} \, \bar{\hat{g}} \right.$$

Trick!
$$\int x f(x) \, f'(x) \, dx = \int (x f(x))^{\wedge} \cdot \overline{(f')^{\wedge}}$$

$$\boxed{(f')^{\wedge}(\xi) = 2\pi i \xi \hat{f}(\xi)}$$

$$= -2\pi i \int (x f(x))^{\wedge}(\xi) \cdot \left(\xi \hat{f}(\xi)\right) d\xi$$

$$f'(x) = 2\pi i \int \underbrace{\xi \hat{f}(\xi)}_{(f')^{\wedge}} e^{2\pi i x\xi} \, d\xi$$

Cauchy Schwartz

$$\leq 2\pi \, \|x f(x)\|_{L^2} \cdot \|\xi \hat{f}(\xi)\|_{L^2}.$$

Also $\int x f(x) f'(x) \, dx = \frac{1}{2} \int x \, \partial_x (f^2) = -\frac{1}{2} \int f^2 = \|f\|_{L^2}^2 = \|f\|_{L^2} \cdot \|\hat{f}\|_{L^2}$

u

Note: $f, g \in S$, $\int f \bar{g} = \int \hat{f} \, \overline{\hat{g}}$. $(*)$

$\mathcal{F}f : L^2 \to L^2 \longrightarrow$ extend $\hat{f}$ from S to L^2.

(using $(*)$).

Q: Is $\int f \bar{g} = \int (\mathcal{F}f) \overline{(\mathcal{F}g)} \quad \forall f, g \in L^2$?

$\mathcal{U}$

$f \in L^2$. Q: What is $\hat{f}(\xi) \longrightarrow \underbrace{\int \hat{f}(\xi)\, e^{-2\pi i \langle x,\xi\rangle}\, dx}$

Officially: defined $\mathcal{F} f$.

$$\mathcal{F} f = \lim_{(L^2)} \hat{f}_n = \lim_{L^2} \int f_n(x)\, e^{-2\pi i \langle x,\xi\rangle}\, dx.$$

like to say $\hat{f}(\xi) = \lim_{R\to\infty} \underbrace{\int_{|x|<R} f(x)\, e^{-2\pi i \langle x,\xi\rangle}\, dx}_{\left(\mathbb{1}_{\{|x|<R\}} f(x)\right)^{\wedge}(\xi)}$ (along a sub seq).

① $\mathbb{1}_{\{|x|<R\}} f \overset{?}{\in} L^1$ (Yes) ✓.

② $\mathbb{1}_{\{|x|<R\}} f(x) \xrightarrow[\text{ptwise, } L^2]{R \to \infty} f(x)$ (DCT).

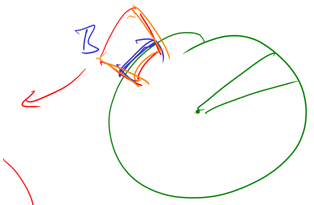
③ $\Rightarrow \hat{f}(\xi) = \lim_{R \to \infty} \int_{|x|<R} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$ (in L^2)

$\Rightarrow \hat{f}(\xi) = \lim_{R \to \infty} \left(\qquad \right)$ for a.e. ξ.

(along a subseq).

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \int_{S^{d-1}} f(ry)\,d\sigma(y)\; r^{d-1}\,dr.$$

$\mathcal{B}$



$$\{A \mid A = \{rx \mid x\in B \ \&\ r\in(a,b)\}\}$$

$\underline{B\subseteq S^{d-1}}$ open.

$\mathcal{G} = \sigma\{A \mid A$ fam $\}$.

Lema 1 : $\mathcal{G} = \mathcal{B}$ (ok).

Lema 2 : $\forall A \in \mathcal{G}$, $\int_A \mathbb{1}_A \, dx = \int_{r=0}^{\infty} \int_{y \in S^{d-1}} \mathbb{1}_A(ry) \, r^{d-1} \, dr \, dy$.

holds ?

Pf: $\mathcal{D} = \{A \in \mathcal{G} \mid$

$\mathcal{D}$ is a σ alg (MC). $\mathcal{D} \supseteq$ a genty set $\Rightarrow \mathcal{D} = \mathcal{G}$.

$A_i \in \mathcal{D}$. $\int$

$A_i \subseteq A_{i+1}$. $A = \bigcup_{1}^{\infty} A_i$. $|A| = \lim |A_i| = \lim \int_0^\infty \int_{S^{d-1}} \mathbb{1}_{A_i}(r y) r^{d-1} \, dr \, dy$

$$\underline{\underline{MC}} \int_0^\infty \int_{S^{d-1}} \mathbb{1}_A (r y) r^{d-1} \, dy \, dr$$

Proof: If $\underline{D}f \in L^2_{per} \Rightarrow f \in H^1_{per}$

$$\underbrace{(Df)^\wedge(n)}_{\Uparrow\ \ell^2} = 2\pi i n \hat{f}(n) \Rightarrow n \hat{f}(n) \in \ell^2$$

$$\Rightarrow f \in H^1_{per}.$$

$$s > \tfrac{1}{2} + m \Rightarrow H^s \subseteq C^m \subseteq H^m$$

HW 13 Q3:

$\alpha \in [0, d]$ $\quad A \in \mathcal{B}(\mathbb{R}^d)$ $\quad H_\alpha(A) < \infty$.

$$\lim_{r \to 0} \frac{H_\alpha(A \cap B(x, r))}{r^\alpha} = 0 \quad \tilde{\forall}\, x \notin A$$

Pf: ① Pick $\varepsilon > 0$. Write $A = K \cup \underline{\underline{B}}$, K cpt $H_\alpha(B) < \varepsilon$.

② $\forall x \notin K$, $\displaystyle\lim_{r \to 0} \frac{H_\alpha(K \cap B(x, r))}{r^\alpha} = 0$

③ $\tilde{\forall} x \notin A$ NTS $\displaystyle\lim_{r \to 0} \frac{H_\alpha(B \cap B(x, r))}{r^\alpha} = 0$

Claim $\left\{ x \;\middle|\; \sup_{r>0} \frac{H_\alpha(B \cap B(x;r))}{r^\alpha} > \lambda \right\} \leq \frac{C}{\lambda} \overbrace{H_\alpha(B)}^{\varepsilon}$

Pf: (Vitali + same pf as the weak L^1 bd of the maximal fn).

Claim $\Rightarrow$ QED.