Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

c_α

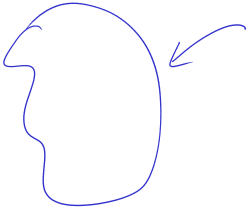
$$H^*_{\alpha,\delta}(A) = \inf\Big\{ \sum_1^\infty \rho_\alpha(E_i) \;\Big|\; \operatorname{diam}(E_i) < \delta\,,\ \text{and } A \subset \bigcup_1^\infty E_j \Big\}, \quad \text{where} \quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta\to 0} H^*_\alpha$.

Proposition A.3 (From homework 2)**.** *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*



A

$$\lambda(A) = \lambda^*(A) = \inf\Big\{ \sum |I_k| \;\Big|\; \bigcup I_k \supseteq A,\ I_k \text{ cells} \Big\}.$$

$$H_d(A) = \lim_{\delta \to 0}\big(\text{sets of diam} \leq \delta\big)$$

Lemma A.5 (Infinite version of Vitali's Covering Lemma)**.** *Let* $W \subseteq \cup_{\alpha \in A} B(x_\alpha, r_\alpha)$*, with* $\boxed{\sup r_\alpha < \infty.}$ *There exists a countable set* $\mathcal{I} \subseteq A$ *such that:*

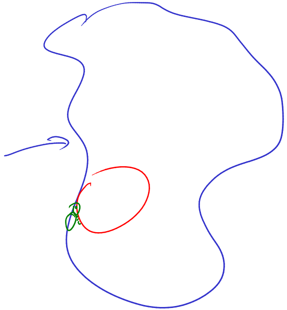
(1) $\{B(x_i, r_i) \mid i \in \mathcal{I}\}$ *are pairwise disjoint.* (countable)

(2) $W \subseteq \cup_{i \in \mathcal{I}} B(x_i, 5r_i)$ *and hence* $|W| \leqslant 5^d \sum_{i \in S} B(x_i, r_i)$*.*

(Similar to finite Vitali Pf).

Lemma A.6. *Let $U \subseteq \mathbb{R}^d$ be open and $\delta > 0$. There exists countably many $x_i \in U$, $r_i \in (0, \delta)$ such that $\overline{B(x_i, r_i)} \subseteq U$, are pairwise disjoint, and $|U - \cup \overline{B(x_i, r_i)}| = 0$.*

$$U = \bigcup_{i=1}^{\infty} B(x_i, r_i)$$



U.

Q: $U = \bigcup_{i=1}^{\infty} \underbrace{B(x_i, r_i)}_{}$ disj?

Claim: Yes except for a null set!

Pf: Lemma: Fix $\delta > 0$.

$\exists\, \alpha < 1$ & $B(x_1, r_1) \cdots B(x_n, r_n) \subseteq U$ $\ni$ $r_i \in (0, \delta)$.

$$\left| U - \bigcup_{i}^{n} \overline{B(x_i, r_i)} \right| < \alpha\, |U| \quad \forall\, U \subseteq \mathbb{R}^d, \text{ open } |U| < \infty.$$

Pf of Lemma A.6: ① Say $|U| < \infty$.

② Set $U_0 = U$. $\quad U_{n+1} = U_n - \bigcup_{i}^{m_n} \overline{B(x_{n,i}, r_{n,i})}$ (from Lemma).

$$|U_{n+1}| < \alpha\, |U_n| \Rightarrow |U_n| < \alpha^n\, |U_0| \xrightarrow[n \to \infty]{} 0.$$

③ Let $N = \bigcap_{i}^{\infty} U_n$. Then $U = N \cup \left(\bigcup_{n,i} \overline{B(x_{n,i}, r_{n,i})} \right)$.

Null! pairwise disj (by constr)

<u>Pf of Lemma:</u> ① $U = \bigcup_{x \in U} B(x, r_x)$ & choose $r_x < \delta$

② Vitali $\Rightarrow \exists\ B(x_i, r_i)$ clsd + pairwise disj +

$$U \subseteq \bigcup B(x_i, 5r_i) \Rightarrow |U| \leq \sum \underbrace{|B(x_i, r_i)|}\cdot 5^d$$

$$\Rightarrow \frac{1}{5^d}|U| \leq \left|\bigcup_1^\infty B(x_i, r_i)\right|$$

For N large, $\bigcup_{1}^{N} B(x_i, r_i) \geq \frac{1}{6^d} |U|$.

Set $V = U - \bigcup_{1}^{N} \overline{B(x_i, r_i)}$.

Note $|V| \leq \left(1 - \frac{1}{6^d}\right) |U|$. QED.

Lemma A.7. $H_d \leqslant \lambda$.

Pf: Fix $\delta > 0$, $U \subseteq \mathbb{R}^d$ open.

$$H_{d,\delta}(U) = H_{d,\delta}\Big(\underbrace{N}_{\lambda(N)=0} \cup \big(\underbrace{\bigcup_{i=1}^{\infty} B(x_i, r_i)}_{\text{disj } \& \ r_i \leq \delta}\big)\Big) \quad (\text{by lemma})$$

$$\leq H_{d,\delta}(N) + \sum_{i=1}^{\infty} H_{d,\delta}\big(\overline{B(x_i, r_i)}\big)$$

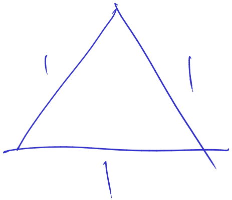
$$\leq H_{d,\delta}(N) + \sum_{i=1}^{\infty} \big|\lambda\big(\overline{B(x_i, r_i)}\big)\big| = \underbrace{H_{d,\delta}(N)}_{=0 \text{ (You check)}} + |U|.$$

$$\Rightarrow \lim_{\delta \to 0} : H_d(U) \leq |U|.$$

Theorem A.8 (Isodiametric inequality)**.** $|A| \leqslant |B(0,1/2)|\,\mathrm{diam}(A)^d = |B(0,\mathrm{diam}(A)/2)|$.

Remark A.9. Note A need not be contained in a ball of radius $\mathrm{diam}(A)/2$.

Eg $A =$



$\mathrm{diam}(A) = 1$

But $A \notin B(x, \tfrac{1}{2}) \quad \forall x \in \mathbb{R}^2$.

Proof of Theorem A.4. $\left(\mathcal{H}_d = \lambda \right)$.

↳ Already know $\mathcal{H}_d \leq \lambda$.

Reverse: Cover U by sets of diam $\leq \delta$.

$$U \subseteq \bigcup_{1}^{\infty} E_k. \Rightarrow |U| \leq \sum |E_k| \underset{\boxed{\text{iso diam}}}{\leq} \sum e_d(E_k)$$

inf over all covers $\Rightarrow |U| \leq \mathcal{H}_{d,\delta}(U)$ send $\delta \to 0$

QED.

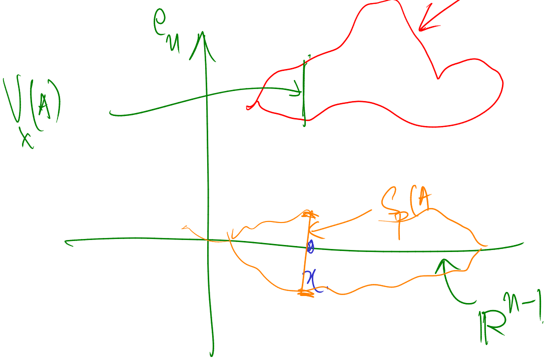
Proposition A.10 (Steiner Symmetrization). *Let $P \subseteq \mathbb{R}^d$ be a hyperplane with unit normal $\hat{n}$. Let $A \in \mathcal{L}(\mathbb{R}^d)$. There exists $S_P(A) \in \mathcal{L}(\mathbb{R}^d)$ such that:*

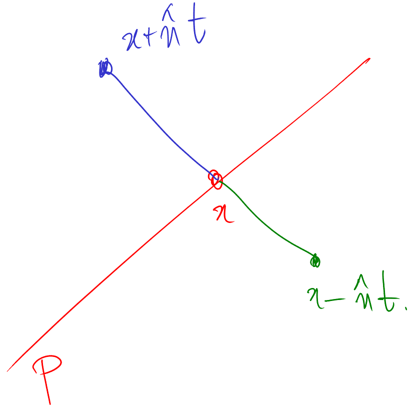
(1) $S_P(A)$ *is symmetric about P (i.e. for any $x \in P$, $t \in \mathbb{R}$, we have $x + t\hat{n} \in \cancel{A}\ S_P(A) \iff x - t\hat{n} \in \cancel{A}\ S_P(A)$).*

(2) $\operatorname{diam}(S_P(A)) \leqslant \operatorname{diam}(A)$.

(3) $|S_P(A)| = |A|$.

Pf/ WLOG. $P = e_n^{\perp}$.





② fix $x \in \mathbb{R}^{n-1}$ $\qquad V_x(A) = \{ y \mid (x,y) \in A \}$.

Set $S_p(A) = \left\{ (x,y) \;\middle|\; x \in \mathbb{R}^{n-1},\ |y| \leq \frac{1}{2} |V_x(A)| \right\}$.

③ Clearly $S_p(A)$ is symm w.r.t. $e_n^{\perp}$.

④ $|S_p(A)| = |A|$ (Fubini / Tonelli)

⑤ $\operatorname{diam}(S_p(A)) \leq \operatorname{diam}(A)$ (pf by picture) (You check).

Proof of Theorem A.8 (NTS. $|A| \leq \left| B\left(0, \frac{\operatorname{diam}(A)}{2}\right) \right|$.

Pf: Let $B = S_{e_1^\perp}\left(S_{e_2^\perp}\left(\cdots S_{e_n^\perp}(A)\right)\right)$

B is symm about all coordinate axis.

$\Rightarrow x \in B \iff -x \in B.$ $\quad (\Rightarrow \operatorname{diam}(B) \geq 2|x|)$

$\Rightarrow B \subseteq B\left(0, \frac{\operatorname{diam}(B)}{2}\right) \Rightarrow |B| \leq \left| B\left(0, \frac{\operatorname{diam}(B)}{2}\right) \right|$

$\Rightarrow |A| \leq B\left(0, \frac{\operatorname{diam}(A)}{2}\right)$ QED.