Theorem 12.26 (Sobolev embedding). *If $s > d/2$ then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous.*

Recall : $\|f\|_{H^s} = \int |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi$, $H_s = \{ f \in L^2 \mid \|\hat{f}\|_{H^s} < \infty \}$

$(s \geq 0)$.

Last time : $f \in S$.

$$\|f\|_{L^\infty} \underset{\text{inversion}}{\leq} \int_{\mathbb{R}^d} |\hat{f}| = \underbrace{\int (1+|\xi|^2)^{s/2} |\hat{f}(\xi)|}\cdot \frac{1}{(1+|\xi|^2)^{s/2}} \, d\xi .$$

$$\overset{C.S.}{\leq} \|f\|_{H^s} \underbrace{\left(\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^s} \, d\xi \right)^{1/2}}.$$

(Note $\int_{|x|>1} \frac{1}{|x|^{d+\varepsilon}}\, dx < \infty \quad \forall \varepsilon > 0 \quad (x \in \mathbb{R}^d)$)

$$\boxed{\Rightarrow \|f\|_{L^\infty} \leq C \|f\|_{H^s}} \quad (*) \qquad \forall f \in \mathcal{S} \quad (\text{provided } s > \tfrac{d}{2}).$$

(actually true $\forall f \ni f, \hat{f} \in L^1$)

Now given $f \in H^s$, Choose a smooth Approx Id, φ_n.

$\rightarrow$ (1) $\varphi_n * f \longrightarrow f$ in H^s $\quad$ ($\because$ $\hat{\varphi}_n(\xi) \leq 1$

& $\hat{\varphi}_n(\xi) \xrightarrow[n\to\infty]{} 1$

$(\varphi_n * f)^\wedge(\xi) = \hat{\varphi}_n(\xi) \cdot \hat{f}(\xi)$ & DCT)

ERROR: $\varphi_n * f$ need not $\in L^1$. Instead choose $g_n \in \mathcal{S} \ni$ $g_n \longrightarrow f$ in H^s & replace $(\varphi_n * f)$ with g_n in the rest of this proof.

② $\Rightarrow$ $\varphi_n * f$ is Cauchy in H^s

③ $\Rightarrow$ $\varphi_n * f$ is Cauchy in L^∞ ($\because$ by ⊛)

④ $\Rightarrow$ $\underline{\varphi_n * f} \xrightarrow{L^\infty} f$ ($\Rightarrow \exists g$ cts + $g = f$ a.e.)

(H^s ... $\Rightarrow H^s \subseteq L^\infty$ for $s > d/2$.)

⑤ Incl map is cts $\because$ ⊛ holds $\forall f \in S \cap H^s \underset{\text{dense}}{\subseteq} H^s$

QED.

Corollary 12.27. *If $s > n + d/2$, then $H^s(\mathbb{R}^d) \subseteq C_b^n(\mathbb{R}^d)$ and the inclusion map is continuous.*

$C_b^n = \{ f \in C^n(\mathbb{R}^d) \mid f$ & all n^{th} ord derivatives are bdd $\}$

Pf: ① Say $n = 1$. $\boxed{f \text{ nice.}}$

$\| \partial_i f \|_{L^\infty} \leq C \| \partial_i f \|_{H^{s-1}}$ (Embedding $\because s - 1 > \frac{d}{2}$)

$\leq C \|f\|_{H^s}$

② $\partial_i(\varphi_n * f)$ Cauchy in H^s $\Rightarrow$ Cauchy in L^∞ $\Rightarrow$ $\partial_i(\varphi_n * f) \xrightarrow{L^\infty} g_i$

Integrate in x_i & get $g_i = \partial_i f$. $\Rightarrow f \in C^1$

QED.

Proposition 12.28 (Elliptic regularity)**.** *Say $f \in \mathcal{S}(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ is such that $\underbrace{\lim_{|x|\to\infty} |x|^d |\nabla u(x)| = 0}$ and $\boxed{-\Delta u = f}$, then $\boxed{u \in \mathcal{S}}$.*

C_c^∞

Note: only need $u \in C^2$ to make sense of $-\Delta u = f$.

$$\Delta u = \sum_{i=1}^{d} \partial_i^2 u.$$

Pf: $-\Delta u = f \Rightarrow -(\Delta u)^\wedge = \hat{f}$

$$\Rightarrow -\left(\sum \partial_j^2 u\right)^\wedge(\xi) = \hat{f}(\xi) \Rightarrow +4\pi^2 |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{4\pi^2 |\xi|^2}.$$

$f \Rightarrow \hat{u}(\xi)$ decays 2 degrees faster than $\hat{f}(\xi)$ as $|\xi| \longrightarrow \infty$

$$\Rightarrow \text{If } \int_{\underline{|\xi|>1}} (1+|\xi|^2)^{2s} |\hat{f}(\xi)|^2 < \infty$$

$$\Rightarrow \int_{\underline{|\xi|>1}} (1+|\xi|^2)^{2(s+2)} |\hat{u}(\xi)|^2 \, d\xi < \infty .$$

How about $\int_{|\xi|<1}$:   Recall $\hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{4\pi^2 |\xi|^2}$.

$$\left.\begin{array}{ll} \text{Obs 1:} & \hat{f}(0) = 0 \\ \text{Obs 2:} & \partial_j \hat{f}(0) = 0 \\ \text{Obs 3:} & \hat{f} \in C^2. \end{array}\right\} \Longrightarrow \frac{\hat{f}(\xi)}{|\xi|^2} \text{ remains bdd as } \xi \longrightarrow 0.$$

Pf of 1: Know $-\Delta u = f$ $\Rightarrow$ $\displaystyle \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} (-\Delta u) = \lim_{R\to\infty} -\int_{|x|<R} \Delta u$

$$= \lim_{R\to\infty} -\int_{|x|<R} \nabla \cdot \nabla u.$$

div thm $\displaystyle = \lim_{R\to\infty} \int_{\partial B(0,R)} \nabla u \cdot \hat{n} \, d\sigma \longrightarrow 0$ by assumption $\Rightarrow$ Obs 1.

<u>Obs 2:</u> $-\Delta u = f \Rightarrow -x_j \Delta u = x_j f$.

Int & use div thm get $\displaystyle\int_{\mathbb{R}^d} \underbrace{x_j f(x)}\, dx = 0$

$\Rightarrow \partial_j \hat{f}(0) = 0 \Rightarrow$ Obs 2.

QED.

not on final

Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

$$H^*_{\alpha,\delta}(A) = \inf\Big\{ \sum_1^\infty \rho_\alpha(E_i) \;\Big|\; \operatorname{diam}(E_i) < \delta\,, \text{ and } A \subset \bigcup_1^\infty E_j \Big\}\,, \quad \text{where} \quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1 + \frac{\alpha}{2})} \Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0, r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta\to 0} H^*_\alpha$.

Proposition A.3 (From homework 2)**.** *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*