Last time: $f \in L^1$, $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle} dx$

$$\langle f, g \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \quad \forall f, g \in \mathcal{S}$$

$$\langle f, g \rangle = \int f \bar{g} \quad \Rightarrow \quad \|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

$\Rightarrow$ Let $\mathcal{F}f = \hat{f}(\xi)$. $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ is an L^2 isom $\Rightarrow \mathcal{F}$ extends to an isom on L^2

Definition 12.20. Let $s \geqslant 0$ and define the *Sobolev space of index s* by

$$H^s = \{f \in L^2(\mathbb{R}^d) \mid \|f\|_{H^s} < \infty\}, \qquad \text{where} \qquad \|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}.$$

Remark 12.21. A function $f \in H^1$ if and only if f and all first order *weak derivatives* are in L^2.

Remark 12.22. For $s < 0$, one needs to define H^s as the completion of $\mathcal{S}$ under the H^s norm.

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2}$$

Proposition 12.23. *Let $s \in (0,1)$. Then $f \in H^s$ if and only if* $\int_0^\infty \left(\frac{\|f - \tau_{hv} f\|_{L^2}}{|h|^s}\right)^2 \frac{dh}{h} < \infty$ *for all $v \in \mathbb{R}^d$.*

Remark 12.24. For $s = 1$, we instead need $\sup_{h>0} \frac{1}{h} \|f - \tau_{hv} f\|_{L^2} < \infty$.

Remark 12.25. If $s \in (0,1]$, then there exists $C = C(s)$ such that $\|f - \tau_h f\|_{L^2} \leqslant C|h|^s \|f\|_{L^2}$ for all $f \in H^s$, $h \in \mathbb{R}^d$.

$s \in (0,1)$. $f \in H^s(\mathbb{R}^d) \iff \int_0^\infty \left(\frac{\|f - \tau_h f\|_{L^2}}{|h|^s}\right)^2 \frac{dh}{h^d}$

Note: $C^\alpha = \{ f \mid \sup \frac{|f(x) - f(y)|}{|x-y|^\alpha} + |f(x)| < \infty \}$
$(\alpha \in (0,1))$

$C^\alpha = \{ f \in L^\infty \mid \|f - \tau_x f\|_{L^\infty} \leq C|x|^\alpha \ \forall x \in \mathbb{R}^d \}$

$(s < 1)$ $H^s \neq$ Prop $\Rightarrow \forall f \in H^s$, $\|f - \tau_h f\|_{L^2} \leq o(|h|^s)$

Prop: Say $f : [0,\infty) \to \mathbb{R}$.

$$\int_{x\in\mathbb{R}^d} f(|x|)\,dx = c_d \int_{r=0}^{\infty} f(r)\, r^{d-1}\, dr$$

$c_d =$ 'surface area' of $S^{d-1} \hookrightarrow \mathbb{R}^d$.

$\left(S^{d-1} = \{x\in\mathbb{R}^d \mid |x|=1\}\right)$

Pf: ① Say $f \in H^s(\mathbb{R}^d)$ $s\in(0,1)$

WTS $$\int_{\mathbb{R}^d} \left(\frac{\|f-\tau_h f\|_2^2}{|h|^{2s}}\right)\frac{dh}{|h|^d} < \infty$$

Note $$\int_{\mathbb{R}^d} \frac{\|f-\tau_h f\|_{L^2}^2}{|h|^{2s}} \frac{dh}{|h|^d} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|1-e^{-2\pi i\langle h,\xi\rangle}|^2}{|h|^{2s}} |\hat{f}(\xi)|^2 \, d\xi \, \frac{dh}{|h|^d}.$$

(Choose δ shortly)

$$\leq \int_{\xi\in\mathbb{R}^d} |\hat{f}(\xi)|^2 \Bigg(\underbrace{C\int_{|h|<\delta} \frac{|h|^2|\xi|^2}{|h|^{2s}} \frac{dh}{|h|^d}}_{①} + \underbrace{\int_{|h|\geq\delta} 2 \frac{dh}{|h|^{d+2s}}}_{②} \Bigg) \quad (*)$$

A better way to do this (courtsey Ethan) is to change variables and replace h with $h/|\xi|$

$$① : \int_{|h|<\delta} |\xi|^2 \frac{dh}{|h|^{d+2s-2}} = C_d \int_{r=0}^{\delta} |\xi|^2 \frac{r^{d-1}\,dr}{r^{d+2s-2}} = C_d |\xi|^2 \int_0^\delta \frac{dr}{r^{2s-1}}$$

while $2s-1<1 \Rightarrow c_d |\xi|^2 \int_0^\delta \frac{dr}{r^{2s-1}} = C|\xi|^2 \Big[r^{2-2s} \Big]_0^\delta$

$$= C \frac{|\xi|^2}{2s-2} \delta$$

② $\displaystyle\int_{|h|>\delta} \frac{dh}{|h|^{d+2s}} = \int_{r=\delta}^{\infty} \frac{dr}{r^{2s+1}} = \int_{r=\delta}^{\infty} r^{-1-2s}\, dr = \left[\frac{r^{-2s}}{(-2s)} \right]_\delta^\infty$

$$= \frac{1}{\delta^{2s}}$$

From (*).

$$\int_{\mathbb{R}^d} \frac{\|\tau_h f - f\|_{2}^{2}}{|h|^{2s}} \frac{dh}{|h|^d} \leq C \int_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)|^2 \left(\frac{|\xi|^2}{\delta^{2s-2}} + \frac{1}{\delta^{2s}} \right) d\xi.$$

Choose δ s.t. $|\xi|^2 = \delta^{-2}$. $\Longleftrightarrow$ $\delta = \frac{1}{|\xi|}$

$$\leq C \int_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)|^2 \left(2 |\xi|^{2s} \right) \leq \|f\|_{H^s}^2 < \infty$$

Q.E.D.

Theorem 12.26 (Sobolev embedding). *If $s > d/2$ then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous.*

$C_b(\mathbb{R}^d) = L^\infty \cap C(\mathbb{R}^d)$.

Pf: Obs 1: If Inversion holds & $\hat{f} \in L^1 \Rightarrow f$ is cts. (DCT)

$$f(x) = \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)$$

Obs 2: $f \in H^s$ & $s > d/2 \Rightarrow \hat{f} \in L^1$

Pf: $$\int |\hat{f}(\xi)| = \int (1+|\xi|^2)^{s/2} |\hat{f}(\xi)| \cdot \frac{1}{(1+|\xi|^2)^{s/2}} \, d\xi.$$

Cauchy Schwarz

$$\le \underbrace{\left\| (1+|\xi|^2)^{s/2} \hat{f}(\xi) \right\|_{L^2}}_{\|f\|_{H^s}} \cdot \underbrace{\left(\int \frac{1}{(1+|\xi|^2)^s} \, d\xi \right)^{1/2}}_{\left(c_d \int_{r=0}^{\infty} \frac{r^{d-1} \, dr}{\underbrace{(1+r^2)}^{s}} \right)^{1/2}}$$

Note $$\frac{r^{d-1}}{(1+r^2)^s} \approx \frac{1}{r^{2s-d+1}} \quad (r \text{ large})$$

$$s > \frac{d}{2} \iff \underbrace{2s-d+1 > +1} \Rightarrow \int_{r=0}^{\infty} \frac{dr \, r^{d-1}}{(1+r^2)^s} < \infty$$

$$\Rightarrow \int |\hat{f}(\xi)| < \infty \quad \Rightarrow \text{done}$$

QED

$\boxed{\mu \perp \lambda}$

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = 0 \quad \lambda \text{ a.e.}$$