Last time:

Inversion: $f, \hat{f} \in L^1 \Rightarrow f(x) = \int e^{+2\pi i \langle x, \xi \rangle} \hat{f}(\xi)\, d\xi$

Lemma 12.13. *If $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} \, d\xi$.*

Last time: (1) $f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi$.

$\varphi(x) = G_1(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$

$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right)$

$\Big(f(0) = \lim f * \varphi_\varepsilon(0) = \lim \int f(x)\, \varphi_\varepsilon(-x)$

$\tau_x f(y) = f(y-x)$

$= \lim_{\varepsilon \to 0} \int f(x)\, \hat{\hat{\varphi}}_\varepsilon(x)\, dx = \lim_{\varepsilon \to 0} \int \hat{f}(x)\, \underbrace{\hat{\varphi}(\varepsilon x)}_{\downarrow\, 1}\, dy$

$\overset{DCT}{=} \int \hat{f}(x)\, dx$ ✓

(2) $f(x) = (\tau_{-x} f)(0)$

$\overset{(1)}{=} \int_{\mathbb{R}^d} (\tau_{-x} f)^\wedge(\xi)\, d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{+2\pi i \langle x, \xi \rangle}\, d\xi$

QED.

Proof of Theorem 12.10. Only assume $f \in L^1$, $\hat{f} \in L^1$. NTS $f(x) = \int \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$.

$$\left(\varphi_\varepsilon = \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}\right) \right).$$

Note
$$f(x) \underset{(\text{subseq})}{\overset{a.e}{=}} \lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x)$$

$$= \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} (f * \varphi_\varepsilon)^\wedge(\xi) \, d\xi \qquad \left(\because f * \varphi_\varepsilon \in C \cap L^1 \;\&\; (f * \varphi_\varepsilon)^\wedge = \underset{L^1}{\hat{f}} \cdot \underset{L^\infty}{\hat{\varphi}_\varepsilon} \in L^1 \right)$$

$$= \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi) \underbrace{\hat{\varphi}(\varepsilon \xi)}_{\downarrow\, 1} \, d\xi.$$

$$\overset{DCT}{=} \lim_{\varepsilon \to 0} \int e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi) \, d\xi \qquad 1 \quad (\& \ |\hat{\varphi}(\varepsilon \xi)| \le 1)$$

QED.

Remark 12.14. If $f, \hat{f} \in L^1$, then $\|f - \varphi_\varepsilon * f\|_{L^\infty} \leqslant \|\hat{f} - (\varphi_\varepsilon * f)^\wedge\|_{L^1} \to 0$

Remark 12.15. If $f, \hat{f} \in L^1$ then $\hat{\hat{f}}(x) = f(-x)$.

$\Rightarrow f, \hat{f} \in L^1 \Rightarrow \exists g \ni f = g$ a.e. & g is cts.

Pf: $f(x) - \varphi_\varepsilon * f(x) = \int \left[\hat{f}(\xi) - (\varphi_\varepsilon * f)^\wedge(\xi)\right] e^{2\pi i \langle x, \xi\rangle} \, d\xi$ (inv).

$\Rightarrow \|f - \varphi_\varepsilon * f\|_\infty \leq \|\hat{f} - (\varphi_\varepsilon * f)^\wedge\|_{L^1} = \|(1 - \hat{\varphi}(\varepsilon\xi))\hat{f}(\xi)\|_{L^1} \xrightarrow[\varepsilon \to 0]{DCT} 0$

$\Rightarrow \varphi_\varepsilon * f \xrightarrow{L^\infty} f$ ✓

$\hookrightarrow$ Let $Rf(x) = f(-x)$

$$(\hat{f})^{\wedge}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{-2\pi i \langle x, \xi \rangle}\, d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i \langle -x, \xi \rangle}\, d\xi \overset{\text{inversion}}{=} f(-x)$$

QED.

12.3. L^2-theory.

Theorem 12.16 (Plancherel). *The Fourier transform extends to a bijective linear isometry on $L^2(\mathbb{R}^d;\mathbb{C})$.*

Note: $f \in L^2$, $\int e^{2\pi i \langle x,\xi\rangle} f(x)\,dx$ may not be defined (in the Lebesgue sense)

($f \in L^2 \not\Rightarrow f \in L^1$ & so $e^{2\pi i\langle x,\xi\rangle} f(x)$ need not $\in L^1(x)$)

Pick $C_c^\infty \in L^2$ (dense). Let $\mathcal{F}f = \hat{f}$ $\forall f \in C_c^\infty$

Claim: $\mathcal{F} : \cancel{C_c^\infty}\ \mathcal{S} \longrightarrow L^2$ is an L^2 isometry! & use this to extend $\mathcal{F}$ to L^2.

Definition 12.17. Define the *Schwartz space*, $\mathcal{S}$, to be the set of all smooth functions such that $\sup_x (1 + |x|^n)|D^\alpha f(x)| < \infty$ for all $n \in \mathbb{N}$ and multi-indexes α.

Remark 12.18. Note $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}$, and so $\mathcal{S}$ is a dense subset of $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

$\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}$. $D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$ (multi-index notation for derivative).

$$\mathcal{S} = \Big\{ f \;\Big|\; \sup_x D^\alpha f(x)\,(1+|x|^n) < \infty \quad \forall \alpha \text{ (multi indexes)} \;\&\; \forall n \geq 0 \Big\}$$

Q: If $f \in C_c^\infty$ does $\hat{f} \in C^\infty$? (Yes $\to$ decay of f $\rightsquigarrow$ diff of $\hat{f}$)

Q: If $f \in C_c^\infty$ does $\hat{f} \in C_c^\infty$? NO!

$\boxed{\text{Q: If } f \in \mathcal{S} \text{ does } \hat{f} \in \mathcal{S}? \leftarrow \text{Yes}}$

Lemma 12.19. *If $f, g \in \mathcal{S}$, then $\int_{\mathbb{R}^d} f\,\bar{g}\,dx = \int_{\mathbb{R}^d} \hat{f}\,\bar{\hat{g}}\,d\xi$.*

(f, g $\mathbb{C}$ valued)

(i.e. $\langle f, g \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d, \mathbb{C})}$ $\forall f, g \in \mathcal{S}$)

Pf: ① $\overline{\hat{g}}(\xi) = \overline{\int g(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx} = \int \overline{g(x)}\, e^{+2\pi i \langle x, \xi \rangle}\, dx$

$= \widehat{\overline{Rg}}(\xi)$ $\quad (Rg(x) = g(-x))$

$$\boxed{\Rightarrow \overline{\hat{g}} = \widehat{\overline{Rg}}}$$

② $\Rightarrow \int \hat{f}\, \overline{\hat{g}}\, d\xi = \int_{\mathbb{R}^d} \hat{f}\, \widehat{\overline{(Rg)}}\, d\xi = \int_{\mathbb{R}^d} \hat{\hat{f}}\, \overline{(Rg)} = \int (Rf)\, \overline{(Rg)}$

Proof of Theorem 12.16

(1) $\forall\, f, g \in \mathcal{S}$, know $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$.

(2) Define $\mathcal{F}f = \hat{f}$ $\forall\, f \in \mathcal{S}$. ($\mathcal{S} \subseteq L^2$ is dense).

(3) $\forall\, g \in L^2$, Pick $f_n \in \mathcal{S} \ni (f_n) \xrightarrow{L^2} g$. Define $\mathcal{F}g = \underbrace{\lim \hat{f}_n}_{L^2 \text{ limit.}}$

(Note f_n is Cauchy in $L^2 \cap \mathcal{S} \Rightarrow \hat{f}_n$ is Cauchy in $L^2 \cap \mathcal{S} \Rightarrow$ lim exists).

(4) $\langle \mathcal{F}f, \mathcal{F}g\rangle = \lim \langle \hat{f}_n, \hat{g}_n \rangle$ $(f_n, g_n \in \mathcal{S})$

$= \int_{\mathbb{R}^d} f g$ QED.

$$= \lim \langle f_n, g_n \rangle = \langle f, g \rangle.$$

$\Rightarrow \mathcal{F}$ is an isometry on L^2.

(5) Note $\mathcal{F} : L^2 \longrightarrow L^2$ is bijective

(Pf: $\mathcal{F}^2 f = Rf \quad \forall f \in \mathcal{S} \Rightarrow \mathcal{F}^2 f = Rf \quad \forall f \in L^2$

$\Rightarrow \mathcal{F}^4 f = f \quad \forall f \in L^2 \Rightarrow$ bij QED).