Last time: $f \in L^1$, $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$

Goal: $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{+2\pi i \langle x, \xi \rangle}\, d\xi$.

(1) $(\tau_x f)^\wedge(\xi) = e^{-2\pi i \langle x, \xi \rangle}\, \hat{f}(\xi)$

(2) $(\partial_j f)^\wedge(\xi) = 2\pi i\, \xi_j\, \hat{f}(\xi)$ & $\partial_j \hat{f}(\xi) = (-2\pi i\, x_j f(x))^\wedge(\xi)$

(3) $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda \xi)$ $\left(\delta_\lambda f(x) = \frac{1}{\lambda^d} f\left(\frac{x}{\lambda}\right) \right)$

Theorem 12.9 (Riemann-Lebesgue Lemma). *If $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

$$C_0 = \{ f \mid f \text{ is cts } \& \lim_{|x|\to\infty} f(x) = 0 \}$$

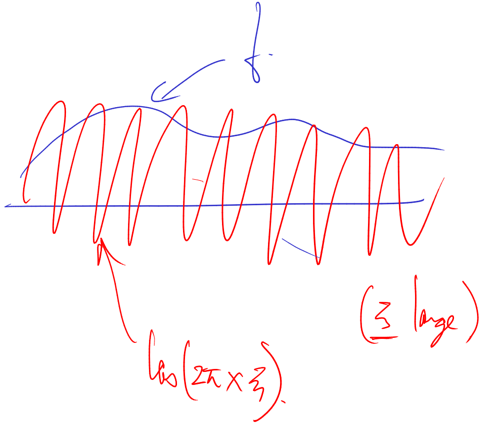
Pf: ① $|\hat{f}(\xi)| = \left| \int f(x) e^{-2\pi i \langle x, \xi \rangle} dx \right| \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_{L^1}$

② $\lim_{h\to 0} \hat{f}(\xi + h) = \lim_{h \to 0} \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi + h, x \rangle} dx$

$$\overset{DC}{=} \int \left(\lim_{h\to 0} f(x) e^{-2\pi i \langle \xi + h, x \rangle} \right) dx = \hat{f}(\xi)$$

$(\because |f(x) e^{i(\ )}| \leq f(x) \in L^1)$

③ NTS $\hat{f}(\xi) \longrightarrow 0$ as $|\xi| \longrightarrow \infty$.



Pf: $(\tau_x f)^\wedge(\xi) = e^{-2\pi i \langle x, \xi \rangle} \hat{f}(\xi)$

$$(f - \tau_x f)^\wedge(\xi) = \Big(1 - \underbrace{e^{2\pi i \langle x, \xi \rangle}}_{-1}\Big) \hat{f}(\xi)$$

$$= 2\hat{f}(\xi).$$

Choose $x = \dfrac{\xi}{2|\xi|^2}$.

$\Rightarrow \quad e^{2\pi i \langle x, \xi \rangle} = e^{i\pi} = -1$

$\Rightarrow$ for $x = \dfrac{\xi}{2|\xi|^2}$, $2\hat{f}(\xi) = (f - \tau_x f)^\wedge(\xi)$

$$\Rightarrow 2|\hat{f}(\xi)| \leq \|f - \tau_x f\|_{L^1} \xrightarrow{|x| \to 0} 0$$

$$\Rightarrow \lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0.$$

QED.

12.2. Fourier Inversion.

Theorem 12.10 (Inversion). *If $f, \hat{f} \in L^1$, then $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle}\, d\xi$.* (for a.e. $x \in \mathbb{R}^d$).

Direct proof attempt:

$$\int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi = \int_{\xi \in \mathbb{R}^d} \left(\int_{y \in \mathbb{R}^d} f(y) e^{-2\pi i \langle y, \xi \rangle} dy \right) e^{+2\pi i \langle x, \xi \rangle} d\xi$$

$$= \int_{\xi \in \mathbb{R}^d} \int_{y \in \mathbb{R}^d} f(y) e^{2\pi i \langle x - y, \xi \rangle} dy\, d\xi. \quad \text{(Can't Fubini!)}$$

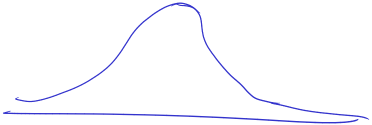
$$\underset{\text{Fubini anyway}}{=} \int_{y \in \mathbb{R}^d} \overbrace{\int_{\xi \in \mathbb{R}^d} f(y) e^{2\pi i \langle x - y, \xi \rangle} d\xi}\, dy$$

$$\underbrace{}_{\delta_x}$$

$$= \int_{y \in \mathbb{R}^d} f(y)\, \delta(x-y)\, dy = f(x) \qquad \text{"QED"}$$

Correct Pf of inversion

Lemma 12.11. *If* $G(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, *then* $\hat{G}(\xi) = e^{-|2\pi\xi|^2/2}$, *and hence* $\boxed{\hat{\hat{G}} = G.}$



Pf: ① Enough to compute $\hat{G}(\xi)$ for $d = 1$

$\left(\because \ \hat{G}(\xi) = \hat{G}(\xi_1)\, \hat{G}(\xi_2) \cdots \hat{G}(\xi_d) \quad \because \ e^{-\sum x_i^2} = \prod e^{-x_i^2} \right)$

② $G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. $\qquad G'(x) = -x\,G(x)$

$$\Rightarrow (G')^{\wedge}(\xi) = -\left(x\,G(x)\right)^{\wedge}(\xi)$$

$$\Rightarrow 2\pi i \xi\, \hat{G}(\xi) = \frac{1}{2\pi i}\left(-2\pi i\, x\, G(x)\right)^{\wedge}(\xi)$$

$$\Rightarrow \quad 2\pi i\,\xi\,\hat{G}(\xi) = \frac{1}{2\pi i}\left(\hat{G}\right)'(\xi)$$

$$\Rightarrow \left(\hat{G}\right)'(\xi) = -\,4\pi^2\xi\,\hat{G}(\xi)$$

$$\Rightarrow \hat{G}(\xi) = \underbrace{\hat{G}(0)}_{1}\,e^{-2\pi^2\xi^2}$$

$$\boxed{\hat{G}(\xi) = e^{-2\pi^2\xi^2}}$$

$$\left(\hat{G}(0) = \int G(x)\,\underbrace{e^{-2\pi i\langle 0,\xi\rangle}}_{1}\,dx = 1\right)$$

(You check $\Rightarrow \hat{\hat{G}} = G$).

Lemma 12.12. *If $f, g \in L^1$ then $\int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}^d} \hat{f} g$.*

Pf:
$$\int_{\mathbb{R}^d} f(x)\,\hat{g}(x)\,dx = \int_{\mathbb{R}^d} f(x) \int_{y\in\mathbb{R}^d} g(y)\, e^{-2\pi i\langle x,y\rangle}\,dy\,dx$$

$$\overset{\text{Fubini}}{=} \int_{y\in\mathbb{R}^d} \underbrace{\int_{x\in\mathbb{R}^d} f(x)\, g(y)\, e^{-2\pi i\langle x,y\rangle}\,dx}\,dy$$

$$= \int_{y\in\mathbb{R}^d} g(y)\cdot\hat{f}(y)\,dy$$

QED.

Lemma 12.13. *If $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then* $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} \, d\xi$.

Pf: ① Prove this for $x = 0$.

i.e. NTS $\int_{\mathbb{R}^d} \hat{f}(\xi) \, d\xi = f(0)$

Let $\varphi(x) = G(x)$, $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}\right)$

Know $\hat{\varphi}_\varepsilon(x) = \hat{\varphi}(\varepsilon x) = \hat{G}(\varepsilon x)$

$$f(0) = \lim_{\varepsilon \to 0} f * \varphi_\varepsilon(0) = \lim_{\varepsilon \to 0} \int f(x) \, \varphi_\varepsilon(-x) \, dx$$

$$= \lim_{\varepsilon \to 0} \int f(x)\, \varphi_\varepsilon(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \int f(x)\, \hat{\hat{\varphi}}_\varepsilon(x)\, dx \qquad \left(\because \hat{\hat{G}} = G \right)$$

$$= \lim_{\varepsilon \to 0} \int \hat{f}(x)\, \hat{\varphi}_\varepsilon(x)\, dx = \lim_{\varepsilon \to 0} \int \hat{f}(x)\, \hat{\varphi}(\varepsilon x)\, dx$$

$$\underline{\underline{DC}} \int \hat{f}(x)\, dx \qquad \left(\begin{array}{l} |\hat{\varphi}(\varepsilon x)| \leq 1 \\ \& \ \hat{\varphi}(\varepsilon x) \xrightarrow{\varepsilon \to 0} \hat{\varphi}(0) = 1 \end{array} \right)$$

QED.