12. **Fourier Transform**

12.1. **Definition and Basic Properties.**

(1) Recall if $f \in L^2_{per}([0,1])$, we set $e_n(x) = e^{2\pi i n x}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx$ and got $f = \sum a_n e_n$ in L^2.

(2) Suppose now $f \in L^2_{per}([-L/2, L/2])$. Can we rescale and send $L \to \infty$?

$$X = \left[-\frac{L}{2}, \frac{L}{2}\right]. \qquad e_n(x) = \frac{e^{-2\pi i n x/L}}{\sqrt{L}}. \qquad \int_{-L/2}^{L/2} |e_n|^2 = 1$$

$$a_n = \langle f, e_n \rangle = \int_{-L/2}^{L/2} f(x) e^{-2\pi i \left(\frac{n}{L}\right) x} \frac{dx}{\sqrt{L}}. \quad \text{Know } \boxed{f(x) = \sum a_n e_n(x)}$$

Let $\xi = \frac{n}{L}$ send $L \to \infty$ & $n \to \infty$ & hold $\xi = \frac{n}{L}$ constant.

F.T.

$$\text{Let } \hat{f}(\xi) = \sqrt{L} \cdot a_n = \int_{-L/2}^{L/2} f(x) e^{-2\pi i \xi x}\, dx \xrightarrow{L \to \infty} \boxed{\int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi}\, dx}$$

Also, $f(x) = \sum a_n e_n(x) = \sum a_n \frac{e^{2\pi i \xi x}}{\sqrt{L}} = \underbrace{\sum \hat{f}(\xi)\, e^{2\pi i x\xi} \cdot \frac{1}{L}}$

$$\xrightarrow{L\to\infty} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

Define. ~~Guess~~: $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$

Guess: $f(x) = \int_{-\infty}^{\infty} \underline{\hat{f}(\xi)\, e^{+2\pi i x\xi}}\, d\xi.$

Definition 12.1. If $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, define the *Fourier transform* of f (denoted by $\hat{f}$) by $\hat{f}(\xi) = \int_{\mathbb{R}^d} \overset{f(x)}{\cancel{\hat{f}(\xi)}} e^{-2\pi i\langle x,\xi\rangle}\, dx$

Remark 12.2. More generally, if μ is a finite (signed) Borel measure, then can define $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle}\, d\mu(x)$.

Analogous to Fourier series, we will show that $\hat{f}$ is defined even for $f \in L^2$, and prove $\boxed{f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{+2\pi i\langle x,\xi\rangle}\, d\xi.}$

$\langle x, \xi \rangle = \sum x_i \xi_i = x \cdot \xi$ (in $\mathbb{R}^d$).

Lemma 12.3 (Linearity). *If $f, g \in L^1$, $\alpha \in \mathbb{R}$ then $(f + \alpha g)^\wedge = \hat{f} + \alpha\hat{g}$.*

Lemma 12.4 (Translations). *Let $\tau_y f(x) = f(x - y)$. Then $(\tau_y f)^\wedge(\xi) = e^{-2\pi i\langle y,\xi\rangle}\hat{f}(\xi)$.*

Lemma 12.5 (Dilations). *Let $\delta_\lambda f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. Then $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$.*

$$(f+\alpha g)^\wedge(\xi) = \int (f+\alpha g)(x)\, e^{-2\pi i\langle x,\xi\rangle}\,dx = \int f(x)\, e^{-2\pi i\langle x,\xi\rangle} + \alpha \int g(x) e^{(\;)}$$

$$= \hat{f}(\xi) + \hat{g}(\xi)$$

Pf: $(\tau_y f)^\wedge(\xi) = \int f(x-y)\, e^{-2\pi i\langle x,\xi\rangle}\,dx = \underbrace{\left(\int f(x-y)\, e^{-2\pi i\langle x-y,\xi\rangle}\,dx\right)}_{\hat{f}} e^{-2\pi i\langle y,\xi\rangle}$

$$= \hat{f}(\xi)\, e^{-2\pi i\langle y,\xi\rangle}$$

Pf: $(\delta_\lambda f)\hat{}(\xi) = \int \frac{1}{\lambda^d} f\left(\frac{x}{\lambda}\right) e^{-2\pi i \langle \frac{x}{\lambda}, \lambda\xi \rangle} dx = \int f(y) e^{-2\pi i \langle y, \lambda\xi \rangle} dy$

$= \hat{f}(\lambda\xi)$ ✓

Lemma 12.6. *If $f, g \in L^1$, then $(f * g)^\wedge = \hat{f}\hat{g}$.*

Pf: Note $f, g \in L^1 \Rightarrow f * g \in L^1$ (Young / ~~Fubi~~ Tonelli)

$$(f*g)^\wedge(\xi) = \int f*g(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx = \int_{x \in \mathbb{R}^d} \int_{y \in \mathbb{R}^d} f(y)\, g(x-y)\, e^{-2\pi i \langle x, \xi \rangle}\, dy\, dx$$

$$= \int_{x \in \mathbb{R}^d} \int_{y \in \mathbb{R}^d} f(y)\, g(x-y)\, e^{-2\pi i \langle x-y, \xi \rangle}\, e^{-2\pi i \langle y, \xi \rangle}\, dy\, dx$$

$$\underset{\text{Fubini \& Let } z = x-y}{=} \hat{f}(\xi)\, \hat{g}(\xi) \qquad \text{QED.}$$

Lemma 12.7. *If* $(1+|x|)f(x) \in L^1(\mathbb{R}^d)$ *then* $\partial_j \hat{f}(\xi) = (-2\pi i x_j f(x))^\wedge(\xi)$.

Lemma 12.8. *If* $f \in C_0^1$, $\partial_j f \in L^1$, *then* $(\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)$.

$\lim_{x\to\infty} |f(x)| = 0$

Decay of $\hat{f}$ $\Longleftrightarrow$ regularity (diff) of f.

diff (regularity) of $\hat{f}$ $\Longleftrightarrow$ decay of f.

① $(1+|x|)f \in L^1$. WTS $\hat{f}$ is diff &

Fix $j \in \{1,\dots,d\}$, $\dfrac{\hat{f}(\xi + h e_j) - \hat{f}(\xi)}{h} = \dfrac{1}{h}\int \left(f(x)\, e^{-2\pi i \langle x, \xi + h e_j\rangle} - f(x)\, e^{-2\pi i \langle x, \xi\rangle}\right) dx$

$\circledast \quad = \int f(x) \left(\dfrac{e^{-2\pi i \langle x, \xi + h e_j\rangle} - e^{-2\pi i \langle x, \xi\rangle}}{h} \right) dx.$

Note $\left| f(x) \cdot \left(\frac{e^{-2\pi i \langle x, \xi + h e_j \rangle} - e^{-2\pi i \langle x, \xi \rangle}}{h} \right) \right| \overset{\text{MVT}}{\leq} 2\pi |x_j| \, |f(x)| \in L^1.$

Hence by $\circledast$, $\lim_{h \to 0} \frac{\hat{f}(\xi + x e_j) - \hat{f}(\xi)}{h} \overset{\text{DCT}}{=} \int_{\mathbb{R}^d} \underbrace{f(x)(-2\pi i x_j)} e^{-2\pi i \langle x, \xi \rangle} dx$

$= (2\pi i x_j f(x))^{\wedge}(\xi)$ ✓

Pf: $f \in C^1_0$, $\partial_j f \in L^1$ Compute $(\partial_j f)^\wedge(\xi)$.

$$(\partial_j f)^\wedge(\xi) = \int_{\mathbb{R}^d} \cancel{f}(\partial_j f)(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$$

$$\overset{\text{By Parts}}{=} + \int_{\mathbb{R}^d} f(x) \left(+ 2\pi i \xi_j\, e^{-2\pi i \langle x, \xi \rangle} \right) dx$$

$$= 2\pi i \xi_j\, \hat{f}(\xi)$$

Q.E.D.

Theorem 12.9 (Riemann-Lebesgue Lemma). *If $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

(i.e. $\hat{f}(\xi) \xrightarrow{|\xi| \to \infty} 0$).