11.3. Change of variables.

Theorem 11.20. *Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi \colon U \to V$ be C^1 and bijective. If $f \in L^1(V)$, then* $\displaystyle\int_V f\,d\lambda = \int_U f \circ \varphi |\det \nabla\varphi|\,d\lambda.$

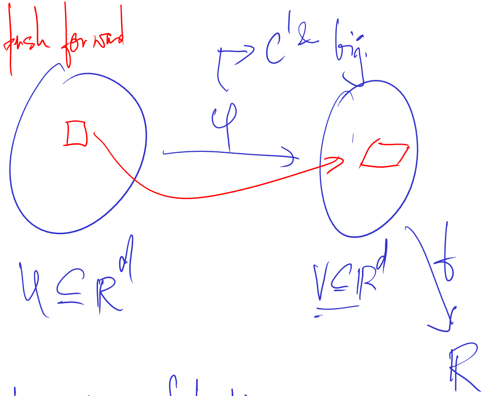
The main idea behind the proof is as follows: Let $\boxed{\mu(A) = \lambda(\varphi(A))}$.

Lemma 11.21. *μ is a Borel measure and $\int_U f \circ \varphi\,d\mu = \int_V f\,d\lambda$.* ← (true from push forward measures)

Lemma 11.22. *$\mu \ll \lambda$*

Lemma 11.23. *$D\mu(x) = |\det \nabla\varphi(x)|$, where $D\mu(x) = \lim_{r \to 0} \dfrac{\mu(B(x,r))}{|B(x,r)|}$.*

Proof of Theorem 11.20. Follows immediately from the above Lemmas. □



$$\int_U f \circ \varphi\, |\det D\varphi|\, d\lambda = \int_V f\, d\lambda$$

Proof of Lemma 11.21

Only NTS μ is a Borel meas. (μ is certainly a meas)

Only NTS $\forall A \in \mathcal{B}(U)$, $\varphi(A) \in \mathcal{B}(V)$.

Note ① $\Sigma = \{A \mid \varphi(A) \in \mathcal{B}(V)\}$ is a σ-alg.

② $\Sigma \supseteq$ all ~~of~~ sets ($\Rightarrow$ all ~~closed~~ & bdd cells)

$\Rightarrow \Sigma \supseteq \mathcal{B}(U)$ QED.

Proof of Lemma 11.22 NTS $\mu \ll \lambda$. Let $A \subseteq U$, $|A| = 0$, NTS $|\varphi(A)| = 0$

ETS $\forall K \subseteq U$ cpt, $\lambda(K) = 0 \Rightarrow |\varphi(K)| = 0$

Say $|K| = 0$. Pick $\varepsilon > 0$, Find $W \supseteq K$ open $\ni$ $|W| < \varepsilon$
& $\overline{W} \subseteq U$ & is cpt.

~~convex~~

Note: $\overline{W}$ cpt $\Rightarrow \sup_{x \in \overline{W}} |\nabla\varphi(x)| = c < \infty$.

$\Rightarrow |\varphi(x) - \varphi(y)| \overset{MVT}{=} |D\varphi(\xi)(x-y)| \leq c|x-y|$ $(\forall x, y \in \Delta)$

(↑ ~~only works if $\overline{W}$ is convex~~).

Δ some convex subset of $\overline{W}$

Can ignore

~~[If $\overline{W}$ is not convex, Cover $\overline{W}$ by N Balls (each cptly contained in U).~~
~~Use the MVT in each ball & get $|\varphi(x)-\varphi(y)| \le N\cdot c\,|x-y|$.)~~

Pick balls $B(x_i, r_i) \ni K \subseteq \bigcup_1^N B(x_i, r_i)$ & $B(x_i, 3r_i) \subseteq \overline{W}$

$$\left|\bigcup_1^N B(x_i, r_i)\right| < \varepsilon \overset{\text{Vitali}}{\Rightarrow} \exists \text{ a disj subset} \ni K \subseteq \bigcup_1^M B(x_i, 3r_i) \text{ \& } \sum_1^M |B(x_i, r_i)| \le 3^d \delta$$

$$\Rightarrow |\varphi(K)| \le \sum_1^{M} |\varphi(B(x_i, 3r_i))| \le \sum_1^{M} c^d\, |B(x_i, 3r_i)|$$

$$\le \delta\, c^d \cdot 3^d.$$

QED.

Proof of Lemma 11.23 NTS $D\mu(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = |\det \nabla \varphi(x)|$

① If $T : R^d \to R^d$ is linear know $|T(A)| = |\det T| \, |A|$

② Pick $x_0 \in U$, Case I : $\nabla \varphi(x_0)$ is inv.

Using ① can W.L. assume $\nabla \varphi(x_0) = I$ (Id matrix).

Also, WL, $x_0 = 0$ & $\varphi(0) = 0$

$\Rightarrow |\varphi(x) - x| < \varepsilon |x| \quad \forall x$ small.

$\Rightarrow \forall r$ small, $\varphi(B(0,r)) \subseteq B(0,(1+\varepsilon)r)$

$$\Rightarrow \overline{\lim_{r\to 0}} \frac{|\varphi(B(0,r))|}{|B(0,r)|} \leq (1+\varepsilon)^d$$

Lower bd: Inv fn thm. φ^{-1} is C^1 (near 0).

Inv contains $\Rightarrow B(0,\frac{r}{1+\varepsilon}) \subseteq \varphi(B(0,r))$

$$\Rightarrow \underline{\lim_{r\to 0}} \frac{|\varphi(B(0,r))|}{|B(0,r)|} \geq \frac{1}{(1+\varepsilon)^d}.$$

ε is arb $\Rightarrow$ $\displaystyle\lim_{r \to 0} \frac{|\varphi(B(0,r))|}{|B(0,r)|} = 1$

QED.

Case 2: $\nabla\varphi(x_0)$ not inv.

(Upper bd pf still works & gives something small).

Please check.