11.2. **Fundamental theorem of calculus.**

Question 11.12. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere imply* $f' \in L^1$*?* NO

Question 11.13. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere, and* $f' \in L^1$ *imply* $f(x) = \int_0^x f'$*?* (NO, Cantor fn)

$$f \in L^1_{loc}(\mathbb{R}^d), \qquad \tilde{\forall} x \in \mathbb{R}^d, \quad f(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

$$\forall K \subset \mathbb{R}^d, \quad f \in L^1(K)$$

$$\int_K |f| < \infty$$

Definition 11.14. We say $f : \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^N |x_i - y_i| < \delta \implies \sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

(x_i, y_i) are finitely many disj intervals

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.

Note: Chose $N = 1$

(Eg: Cantor fn is cts but not a.c.)

Theorem 11.16. *Let $f\colon [a,b] \to \mathbb{R}$ be measurable. Then f is absolutely continuous if and only if f is differentiable almost everywhere, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f'$ ~~almost~~ everywhere.*

Proof of the reverse implication of Theorem 11.16

Assume f diff a.e., $f' \in L^1$, & $f(x) = f(a) + \int_0^x f'$.

NTS f is a.c.

Pf: Let $\varepsilon > 0$. $f' \in L^1 \Rightarrow \exists\, \delta > 0 \ni \mu(E) < \delta \Rightarrow \int_E |f'| < \varepsilon$.

Take $(x_1, y_1) \cdots (x_N, y_N)$ disj $\ni \sum_1^N |x_i - y_i| < \delta$.

$$\Rightarrow \sum_1^N |f(x_i) - f(y_i)| = \sum_1^N \left| \int_{x_i}^{y_i} f' \right| \le \int_{\underbrace{\cup (x_i, y_i)}} |f'| < \varepsilon$$

QED.

Lemma 11.17. *If f is absolutely continuous, $\boxed{\text{monotone and injective}}$, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ ~~almost~~ everywhere.*

(W.L.o.g. assume f is inc)

Pf: (1) Let $\mu(A) = |f(A)|$ $(A \in \mathcal{B})$

Q: $A \in \mathcal{B} \Rightarrow f(A) \in \mathcal{B}$ (Yes)

$\Leftarrow$ (f cts inj $\Rightarrow f^{-1}$ cts $\Rightarrow f(A) \in \mathcal{B}$ whenever $A \in \mathcal{B}$)

$\Rightarrow \mu$ is a finite measure.

(2) Claim: $\mu \ll \lambda$

Pf: Say $A \subseteq [a, b]$, $|A| = 0$, NTS $\mu(A) = 0$

ETS $\forall K \subseteq A$ cpt, $\mu(K) = 0$

Pick any $\varepsilon > 0$. Choose δ as in the def of a.c. of f.

$\exists U \supseteq K \ni |U| < \delta$

K cpt $\Rightarrow \exists (x_1, y_1) \cdots (x_N, y_N)$ disj $\ni \sum_1^N |x_i - y_i| < \delta$

$\Rightarrow \mu\left(\bigcup_1^N (x_i, y_i)\right) = \sum_1^N |f(x_i) - f(y_i)| \overset{a.c}{<} \varepsilon \Rightarrow \mu(K) < \varepsilon$

$\Rightarrow \mu(A) = \sup_{K \subseteq A} \mu(K) = 0 \Rightarrow \mu \ll \lambda.$

③ R.N. $\Rightarrow \exists g \in L^1$ s.t. $d\mu = g\, d\lambda$.

$$\left.\begin{array}{l} \Rightarrow \mu([a,x]) = f(x) - f(a) \\ \qquad \| \\ \quad \displaystyle\int_a^x g(y)\, dy \end{array}\right\} \Rightarrow f(x) = f(a) + \int_a^x g(y)\, dy$$

Lebesgue diff $\Rightarrow$ f is diff a.e. & $f' = g$ a.e.

$$\Rightarrow f(x) = f(a) + \int_a^x f' \quad \text{a.e.}$$

QED.

Lemma 11.18. *If f is absolutely continuous and $\boxed{\text{monotone,}}$ then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ almost everywhere.*

Pf: W.L. assume f is inc.

Let $g(x) = f(x) + x$. Clearly g is strictly inc, & a.c.

$\Rightarrow$ FTC holds for g. i.e. $g(x) = g(a) + \int_a^x g'$

$\Rightarrow f(x) = g(a) + \int_a^x g' - x$

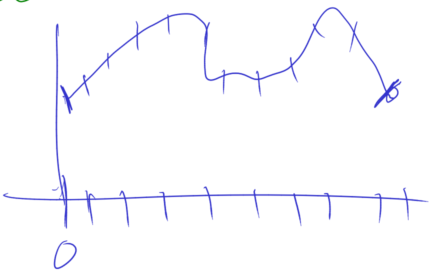
$\Rightarrow f$ is diff a.e. & $f' = g' - 1 \Rightarrow$ QED.

Lemma 11.19. *If f is absolutely continuous then there exist g, h ~~monotone~~ inc such that $f = g - h$.*

Proof of the forward implication of Theorem 11.16. Follows immediately from the previous lemmas. $\square$

Pf: Claim A.C. $\Rightarrow$ f has finite variation. (W.L. $[a,b] = [0,1]$)

$$\mathrm{Var}(f) = \sup \sum |f(x_{i+1}) - f(x_i)|, \text{ where } \{x_0, \dots, x_N\} \text{ is a partition of } [0,1].$$



Note: A.C. $\Rightarrow$ $\mathrm{Var}(f) < \infty$.

Pf: Pick $\varepsilon = 1$. $\exists N$ + $\sum |y_i - x_i| < \frac{1}{N}$

$$\Rightarrow \sum |f(x_i) - f(y_i)| \leq 1.$$

$\boxed{\text{Claim: } \mathrm{Var}(f) \leq N}$

Claim: If f has finite var

$\Rightarrow \exists\, g, h$ inc $\ni$ $f = g - h$.

Pf: Let $F(x) =$ var of f on $[0, x]$

$$= \sup \sum |f(x_{i+1}) - f(x_i)| \quad \text{over all finite part of } [0,x].$$

Clearly : ① F is increasing

(Requires checking) $\rightarrow$ ② $F + f$ & $F - f$ are both increasing.

③ f is a.c. $\Rightarrow$ F is a.c. (immediate)

Hence: If f is a.c.

Define $F(x) =$ var of f on [0, x]

$$\Rightarrow f = \underbrace{\frac{(F+f)}{2}}_{\text{a.c. \& inc}} - \underbrace{\frac{(F-f)}{2}}_{\text{a.c. \& inc.}}$$

$\Rightarrow$ FTC holds for f.

Q.E.D.