Last time: Thm: $f \in L^1(\mathbb{R}^d)$ then $\tilde{\forall} x \in \mathbb{R}^d$

$$f(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

Last time: Vitali. If $W \subseteq \bigcup_{i=1}^{N} B(x_i, r_i)$, then $\exists$ $B(x_{n_1}, r_{n_1}) \ldots B(x_{n_k}, r_{n_k})$ DISJOINT $\Rightarrow W \subseteq \bigcup_{i=1}^{k} B(x_{n_i}, 3r_{n_i})$

$$\left(\Rightarrow |W| \leq 3^d \sum |B(x_{n_i}, r_{n_i})| \right)$$

Definition 11.5 (Maximal function)**.** Let μ be a finite (signed) Borel measure on $\mathbb{R}^d$. Define the *maximal function* of μ by

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}$$

Proposition 11.6. *$M\mu \in L^{1,\infty}$, and $|\{M\mu > \alpha\}| \leqslant \frac{3^d}{\alpha}\|\mu\|$.*

Corollary 11.7. *If $f \in L^1(\mathbb{R}^d)$, then $|\{Mf > \alpha\}| \leqslant \frac{3^d}{\alpha}\|f\|_{L^1}$.*

$f \in L^1(\mathbb{R}^d)$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f|$$

Want $f \in L^1 \Rightarrow Mf \in L^1$ & $\|Mf\|_{L^1} \leq C\|f\|_{L^1}$

(Turns out this is false)

$f \in L^1 \Rightarrow \forall \alpha \quad |\{|f| > \alpha\}| \leq \frac{\|f\|_{L^1}}{\alpha} C$

Claim: $\forall p \in (1,\infty]$ $\exists C_p$ + $\|Mf\|_p \leq C_p \|f\|_p$

In

Pf of Prop: $\mu \to$ finite signed measure, $\alpha > 0$ } Want $|\{M\mu > \alpha\}| \leq \frac{C\|\mu\|}{\alpha}$.

① WLOG assume μ is +ve.

② Let $\alpha > 0$. Pick $K \subseteq \{M\mu > \alpha\}$ cpt. $\left(M\mu(x) = \sup_{r>0} \frac{|\mu(B(x,r))|}{|B(x,r)|} \right)$

$\Rightarrow \forall x \in K, \exists r_x$ st $\mu(B(x, r_x)) > \alpha |B(x, r_x)|$ ... (*)

K cpt $\Rightarrow \exists x_1 \dots x_N$ st $K \subseteq \bigcup_{1}^{N} B(x_i, r_{x_i})$

Vitali $\Rightarrow \exists x_1 \dots x_M$ st $K \subseteq \bigcup_{1}^{M} B(x_i, 3r_{x_i})$ & $\{B(x_i, r_{x_i})\}$ are all disj.

Hence $|K| \leq \left| \bigcup_{i}^{m} B(x_i, 3r_{x_i}) \right| \leq 3^d \sum |B(x_i, r_{x_i})|$

$$\overset{(*)}{\leq} \frac{3^d}{\alpha} \sum \mu(B(x_i, r_{x_i}))$$

$$\overset{(\text{disj})}{=} \frac{3^d}{\alpha} \mu\left(\bigcup_{1}^{m} B(x_i, r_i) \right)$$

$$\leq \|\mu\| \, 3^d/\alpha \qquad \text{QED.}$$

Proposition 11.8. *If $f \in L^1(\mathbb{R}^d)$, then* $\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{|y-x|<r} |f(y)-f(x)|\,dy = 0$ *almost everywhere.*

Remark 11.9. This immediately implies Theorem 11.3.

Thm (Lebesgue) $\tilde{\forall} x$, $f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$.

Pf strategy

① Prove thm for nice fns. (e.g. cts fn).

② $\forall f \in L^1$, write $f = g + h$, $g \to$ nice & $h \to$ small in L^1

③ Obtain a unif bound ~~of~~ for h.

an

Pf: Let $\Omega f(x) = \overline{\lim_{r \to 0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f(y)|\, dy$

① Clearly $f \in C_c$, $\Omega f(x) = 0 \quad \forall x$.

② $\forall \varepsilon > 0$, $\exists g \in C_c$ & $h \in L^1$ s.t. $f = g + h$ & $\|h\|_{L^1} < \varepsilon$.

③ $\Omega f(x) = \Omega h(x) = \overline{\lim_{r \to 0}} \frac{1}{|B(x,r)|} \int_{|B(x,r)|} |h(x) - h(y)|\, dy$

$\leq |h(x)| + |Mh(x)|$

$$\Rightarrow \forall \alpha > 0, \quad |\{\Omega_f > \alpha\}| \leq |\{|h(x)| + |Mh(x)| > \alpha\}|$$

$$\leq |\{|h| > \alpha/2\}| + |\{Mh > \tfrac{\alpha}{2}\}|$$

$$\leq \frac{2\|h\|_{L^1}}{\alpha} + \frac{2\cdot 3^d}{\alpha}\|h\|_{L^1} \leq \frac{C}{\alpha}\underbrace{\|h\|_{L^1}}_{\varepsilon.}$$

$$\Rightarrow \forall \alpha > 0, \quad |\{\Omega_f > \alpha\}| \leq \frac{C\varepsilon}{\alpha} \qquad (\varepsilon \text{ is arb})$$

$$\Rightarrow |\{\Omega_f > \alpha\}| = 0 \quad \forall \alpha > 0. \quad \Rightarrow \Omega_f = 0 \text{ a.e.}$$

QED

Corollary 11.10. *If $\mu \ll \lambda$ is a finite signed measure, then the Radon-Nikodym derivative is given by* $\dfrac{d\mu}{d\lambda} = \lim_{r\to 0} \dfrac{\mu(B(x,r))}{|B(x,r)|}$.

Remark 11.11. Will use this to prove the change of variables formula.

By RN $\Rightarrow$ $\exists f \in L^1$ s.t. $d\mu = f\, d\lambda$.

L. diff $\Rightarrow$ $f(x) \overset{a.e.}{=} \lim_{r\to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} f$

$\dfrac{d\mu}{d\lambda}$ (= $f(x)$)

$= \lim_{r\to 0} \dfrac{\mu(B(x,r))}{\lambda(B(x,r))}$.

Let's now deal with the second fundamental theorem of calculus:

Question 11.12. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere imply* $f' \in L^1$*?*

Question 11.13. *Does* $f\colon [0,1] \to \mathbb{R}$ *differentiable almost everywhere, and* $f' \in L^1$ *imply* $f(x) = \int_0^x f'$*?* (No → Cantor fn)

$$\int_a^b f' = f(b) - f(a). \quad (f' \to \text{R. int})$$

No: Eg $f(x) = \begin{cases} 1/\sqrt{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$

Definition 11.14. We say $f\colon \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^N |x_i - y_i| < \delta \implies \sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

Remark 11.15. Any absolutely continuous function is continuous, but not conversely.