Last time: $\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$

Intuition: faster decay of $\hat{f}$ $\approx$ Better regularity of f.

① $f \in L^1 \Rightarrow (\hat{f}(n)) \longrightarrow 0$

② $f \in L^2 \Rightarrow \sum |\hat{f}(n)|^2 < \infty$

③ $f' \in L^2 \Rightarrow \sum \left| (1+|n|)\hat{f}(n) \right|^2 < \infty$ $\longleftarrow$ $(s=1)$

Definition 10.37. For $s \geqslant 0$, let $H^s_{per} \stackrel{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum(1+|n|)^{2s}|\hat{f}(n)|^2$.

H^s = Sobolev space of index s

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

Theorem 10.39 (1D Sobolev Embedding). *If $s > \frac{1}{2}$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

(≈ "s" weak derivatives in L^2)

Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when $s = 1/2$.

Remark 10.41. In d dimensions the above is still true if you assume $\boxed{s > d/2}$.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.

Note: Higher s is $\Rightarrow$ faster decay of $|\hat{f}(n)|$ as $n \to \infty$.

$\to$ Thm + induction $\Rightarrow$ $s > n + \frac{1}{2}$ then $H^s \subseteq C^n_{per}$

(& the incl map is cts)

Pf of 1D Sobolev. $f \in H^s$, $s > \frac{1}{2}$.

Want $f \in C_{per}$ & $\underbrace{\|f\|_\infty \leq C \|f\|_{H^s}}_{\text{IOU.}}$ for some const C

① Will show f is cts.

Note that $f(x) = \sum \hat{f}(n) e^{2\pi i n x}$ in L^2.

(i.e. $\sum \hat{f}(n) e^{2\pi i n x}$ converges in L^2 to f).

Claim: If $f \in H^s$ ($s > 1/2$), then $\sum \hat{f}(n) e^{2\pi i n x}$ conv unif

$(\Rightarrow f$ is cts$)$.

Pf of claim: Weierstrass: Enough to show $\sum |\hat{f}(n)| < \infty$

(Note: $\underline{f \in L^2} \Rightarrow \sum |\hat{f}(n)|^2 < \infty \not\Rightarrow \sum |\hat{f}(n)| < \infty$!).

Note
$$\sum |\hat{f}(n)| = \sum \frac{1}{(1+|n|)^s} (1+|n|)^s |\hat{f}(n)|$$

Cauchy Schwartz
$$\leq \left(\sum \frac{1}{(1+|n|)^{2s}}\right)^{1/2} \left(\sum (1+|n|)^{2s} |\hat{f}(n)|^2\right)^{1/2}.$$

$< \infty$ when $2s > 1$ $\qquad \|f\|_{H^s}$.

$< \infty \Rightarrow$ QED.

② WTS $\|f\|_\infty \leq C \|f\|_{H^s}$.

Pf: $\|f\|_\infty = \left\| \sum \hat{f}(n) e^{2\pi i n x} \right\|_\infty \leq \sum |\hat{f}(n)|$

$$\leq \left(\sum \frac{1}{(1+|n|)^{2s}} \right)^{1/2} \cdot \|f\|_{H^s} \Rightarrow \text{QED}.$$

Theorem 10.43 (1D Sobolev embedding)**.** *If* $s > \frac{1}{2} - \frac{1}{2n}$, *then* $H^s_{per} \subseteq L^{2n}$ *and the inclusion map is continuous.* (Next week HW)

Remark 10.44. The above is true for $\boxed{s = \frac{1}{2} - \frac{1}{p}}$, for some $p \in [1, \infty)$ but our proof won't work.

$$f \in H^s, \quad s > \frac{1}{2} - \frac{1}{2n} \Rightarrow \int_0^1 |f|^{2n} < \infty$$

Why is "H^s" stuff useful?

(Last Q on this weeks HW)

$L^2 \longrightarrow \infty$ dim V.S. $\{\|f\|_{L^2} \leq 1\}$ is not cpt.

$H^1_{per} \subseteq L^2$. Claim: $\{f \in L^2 \mid \|f\|_{H^1_{per}} \leq 1\} \subseteq L^2$ is relatively cpt!

11. Differentiation

11.1. Lebesgue Differentiation.

Theorem 11.1 (Fundamental theorem of Calculus 1). *If f is continuous and $F(x) = \int_0^x f(t)\,dt$, then F is differentiable and $F' = f$.*

Theorem 11.2 (Fundamental theorem of Calculus 2). *If f is Riemann integrable, and $F' = f$, then $\int_a^b f = F(b) - F(a)$.*

Our goal is to generalize these to Lebesgue integrable functions.

Theorem 11.3 (Lebesgue Differentiation). *If $f \in L^1(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^d$ we have $\cancel{\frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f\,d\lambda} = f(x)$.*

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f \, d\lambda \overset{a.e}{=} f(x)$$

Note: If $d=1$, $f \in L^1(\mathbb{R})$, $F = \int_0^x f$, $\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \lim \frac{1}{2h} \int_{x-h}^{x+h} f$

$\forall A \subseteq \mathbb{R}^d$, $|A| = \lambda(A)$

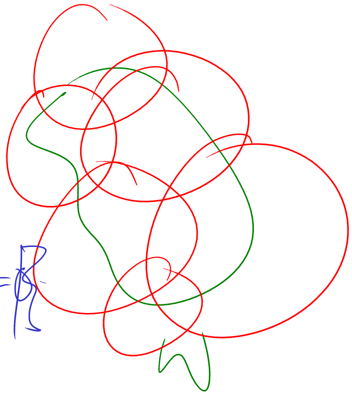
Leb diff $f(x)$ a.e.

Lemma 11.4 (Vitali Covering Lemma). *Let* $W \subseteq \cup_1^N B(x_i, r_i)$. *There exists* $S \subseteq \{1, \ldots, N\}$ *such that:*

(1) $\{B(x_i, r_i) \mid i \in S\}$ *are pairwise disjoint.*

(2) $W \subseteq \cup_{i \in S} B(x_i, 3r_i)$ *and hence* $|W| \leqslant 3^d \sum_{i \in S} B(x_i, r_i)$.

Pf: ① Pick n_0 + $B(x_{n_0}, r_{n_0})$ has the largest radius.



② Pick n_1 + amongst all $\{B(x_i, r_i) \mid B(x_{n_0}, r_{n_0}) \cap B(x_i, r_i) = \emptyset\}$ so that $B(x_{n_1}, r_{n_1})$ has the largest radius.

③ Keep going $r_{n_{k+1}}$ is the largest radius amongst all $B(x_i, r_i)$ that are disj from $B(x_0, r_{n_0}) \cdots B(x_{n_k}, r_{n_k})$

④ Claim this is the desired collection.

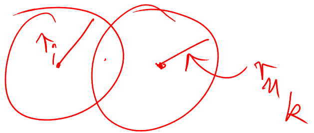
ⓐ clearly $B(x_{n_0}, r_{n_0}), B(x_{n_1}, r_{n_1}) \ldots$ are disj by constr.

ⓑ Pick any $B(x_i, r_i)$ which is not amongst

$$\Rightarrow \exists\, k \ni r_{n_k} \geq r_i \;\&\; B(x_i, r_i) \cap B(x_{n_k}, r_{n_k}) \neq \phi.$$

(by constr)

$$\Rightarrow B(x_{n_k}, 3r_{n_k}) \supseteq B(x_i, r_i)$$



$$\Rightarrow \bigcup_k B(x_{n_k}, 3r_{n_k}) \supseteq \bigcup B(x_i, r_i) \supseteq W$$ QED.

$$\langle S_N f, \cancel{S_N} g \rangle \xrightarrow{\text{Wait}} \langle f, g \rangle$$

$$|\langle S_N f - f, g \rangle| \overset{\text{Holder}}{\leq} \underbrace{\|S_N f - f\|_{L^2}}_{\downarrow\, 0} \|g\|_{L^2} \longrightarrow 0.$$