$X = [0,1]$, $f \in L^1(X)$, $\hat{f}(n) = \langle f, e_n \rangle$.

$$\langle f, g \rangle = \int f \bar{g} \quad , \quad e_n(x) = e^{2\pi i n x}$$

$$S_N f = \sum_{-N}^{N} \hat{f}(n)\, e_n \qquad S_N f = D_N * f$$

(D_N — Dirichlet kernel Not an AI)

$$\sigma_N f = \frac{1}{N} \sum_{0}^{N-1} S_n \qquad \boxed{\sigma_N f = F_N * f}$$

($F_N \to$ Fejer Kernel are an AI)

$\Rightarrow \forall p \in [1,\infty)$, $\underline{\underline{\sigma_N f}} \longrightarrow f$ in L^p.

Orthogonality $\Rightarrow S_N f \longrightarrow f$ in L^2 ($p = 2$!!).

Theorem 10.28. *If $p \in (1, \infty)$, $f \in L^p$ then $S_N f \to f$ in L^p.* $(p \neq 1)$

Proof. The proof requires boundedness of the Hilbert transform and is beyond the scope of this course. $\square$

Theorem 10.29. *If $f \in L^\infty$ and is Hölder continuous at x with any exponent $\alpha > 0$, then $S_n f(x) \to x$.*

Proof. On homework. $\square$

Remark 10.30. If f is simply continuous at x, then certainly $\sigma_n f(x) \to f(x)$, but $S_n f(x)$ need not converge to $f(x)$. In fact, for almost every continuous periodic function function, $S_N f$ diverges on a dense G_δ.

Q: $f \in L^\infty$, Must $S_N f \xrightarrow{L^\infty} f$? (No. $S_N f$ is cts $\forall N$ so if f is not cts $(S_N f) \not\to f$ in L^∞).

Q: $f \in C_{per}([0,1])$ (cts periodic)

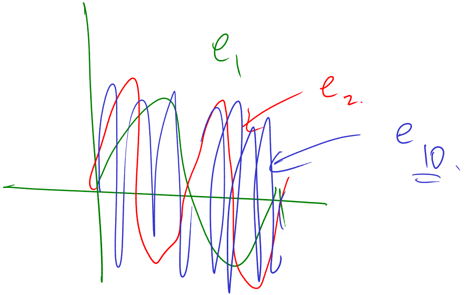
Must $(S_N f) \xrightarrow{L^\infty} f$

The next few results establish a connection between the regularity (differentiability) of a function and decay of its Fourier coefficients.

Theorem 10.31 (Riemann Lebesgue). *Let μ be a finite measure and set $\hat{\mu}(n) = \int_0^1 \overline{e_n}\, d\mu$. If $\mu \ll \lambda$, then $(\hat{\mu}(n)) \to 0$ as $n \to \infty$.*

Theorem 10.32 (Parseval's equality). *If $f \in L^2([0,1])$ then $\|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$.*

Intuition: More diff a fn is $\longrightarrow$ faster decay of Fourier coefficients.



$$\hat{f}(n) = \langle f, e_n \rangle = \int e^{-2\pi i n x} \underbrace{f(x)\, dx}_{d\mu(x)}$$

Q: μ a finite measure

$$(\hat{\mu}) \in \ell^\infty \Longleftarrow \left|\hat{\mu}(n)\right| = \left|\int_0^1 e^{-2\pi i n x}\, d\mu(x)\right| \le \mu([0,1])$$

Pf of Riemann Lebesgue: $\mu \ll \lambda$, By RN, $\exists f \in L^1$ s.t. $d\mu = f\, d\lambda$.

Pick any $\varepsilon > 0$,

1. $\exists N$ s.t. $\| f - \sigma_N f \|_{L^1} < \varepsilon$ $\left(\because (\sigma_N f) \to f \text{ in } L^1 \right)$

$\to$ 2. If $g \in L^1$, $|\hat{g}(n)| \le \|g\|_{L^1}$ $\forall n$

$\Rightarrow \forall n$, $\left| \left(f - \sigma_N f\right)^{\wedge}(n) \right| \le \| f - \sigma_N f \|_1 < \varepsilon$.

3. $\forall n > N$, $(\sigma_N f)^{\wedge}(n) = 0 \Rightarrow$ By 2, $|\hat{f}(n)| < \varepsilon$ $\forall n > N$.

QED.

n

Pf of Parseval: $f \in L^2$. NTS $\|f\|_{L^2} = \|\hat{f}\|_{\ell^2}$

More generally, $f, g \in L^2$, then $\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{\ell^2}$

$$\langle f, g\rangle_{L^2} = \int_0^1 f\bar{g}, \qquad \langle \hat{f}, \hat{g} \rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}$$

Pf: $\langle e_n, e_m \rangle = \delta_{m,n}$

$\Rightarrow \langle S_N f, S_N g \rangle = \sum_{-N}^{N} \hat{f}(n)\overline{\hat{g}(n)}$

$S_N f \xrightarrow{L^2} f$, $S_N g \xrightarrow{L^2} g$ $\overset{\text{Holder}}{\Longrightarrow}$ $\int S_N f \, \overline{S_N g} \longrightarrow \int_0^1 f\bar{g} \Rightarrow \int_0^1 f\bar{g} = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$

QED.

Question 10.33. *What are the Fourier coefficients of f'?*

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)\, e_n(x) = \sum_{-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x}$$

Guess $f'(x) = \qquad = \sum \left(2\pi i n\, \hat{f}(n)\right) e^{2\pi i n x}$

$$\Rightarrow \boxed{(f')^{\wedge}(n) \overset{\text{guess}}{=} 2\pi i n\, \hat{f}(n).}$$

Definition 10.34. We say g is a *weak derivative* of f if $\langle f, \varphi' \rangle = -\langle g, \varphi \rangle$ for all $\varphi \in C^{\infty}_{per}([0,1])$. ($\varphi$ ~~real~~)

Proposition 10.35. *If $f \in L^1$ has a weak derivative $f' \in L^1$, then $(f')^\wedge(n) = 2\pi i n \hat{f}(n)$.*

Corollary 10.36. *If $f \in L^2$ has a weak derivative $f' \in L^2$, then* $\boxed{\sum [(1+|n|)|\hat{f}(n)|]^2 < \infty}$.

If f is ~~diff~~ C^1, $\int_0^1 f \overline{\varphi'} \overset{\text{IBP}}{=} -\int_0^1 \varphi \overline{f'}$

$g \leftarrow$ weak derivative.

Pf: $(f')^\wedge(n) = \langle f', e_n \rangle \overset{\text{IBP}}{=} -\langle f, e_n' \rangle$

$= -2\pi i n \langle f, e_n \rangle = -2\pi i n \hat{f}(n)$

QED.

Note: ① $f \in L^1 \Rightarrow$ R.L. $(\hat{f}(n)) \longrightarrow 0$.

② $f' \in L^1 \Rightarrow ((f')^\wedge(n)) \longrightarrow 0 \Rightarrow (n\hat{f}(n)) \longrightarrow 0$.

Definition 10.37. For $s \geqslant 0$, let $H^s_{per} \stackrel{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum(1+|n|)^{2s}|\hat{f}(n)|^2$. (Sobolev space of order s).

Remark 10.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

Theorem 10.39 (1D Sobolev Embedding). *If $s > \frac{1}{2}$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

Remark 10.40. Need $s > \frac{1}{2}$. The theorem is false when $s = 1/2$.

Remark 10.41. In d dimensions the above is still true if you assume $\boxed{s > d/2}$.

Remark 10.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac{1}{2} + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.