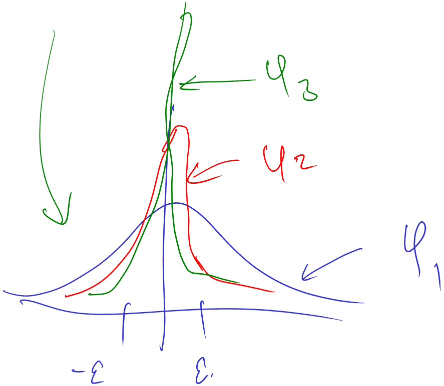
Definition 10.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \geqslant 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 10.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$. $(\varepsilon \to 0)$

Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$, for $x \in \mathbb{R}^d$.



Proposition 10.15. *If $p \in [1, \infty)$, $f \in L^p$, and (φ_n) is an approximate identity, then $\varphi_n * f \to f$ in L^p.*

Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous.

Remk: Useful because (from HW) we know if $\varphi_n \in C_c^\infty \Rightarrow \varphi_n * f \in C^\infty$

Pf of Prop.

$$\varphi_n * f(x) - f(x) = \int_{\mathbb{R}^d} \varphi_n(y)\left[f(x-y) - f(x)\right] dy \quad \left(\because \int \varphi_n = 1\right)$$

$$\Rightarrow \|\varphi_n * f - f\|_p = \left\| \int_{\mathbb{R}^d} \text{———} \, dy \right\|_{L^p} \overset{\text{Minkowski}}{\leq} \int_{y \in \mathbb{R}^d} \varphi(y) \, \|f(\cdot - y) - f(\cdot)\|_{L^p} \, dy$$

Let $\tau_y f(x) = f(x-y)$. $\Rightarrow \|\varphi_n * f - f\|_p \leq \int_{y \in \mathbb{R}^d} \varphi(y) \, \|\tau_y f - f\|_p \, dy$

Pick any $\varepsilon > 0$. $\|\varphi_n * f - f\|_p \leq \int_{|y|<\varepsilon} \varphi_n(y) \underbrace{\|\tau_y f - f\|_p}_{\text{small when } y \text{ is small!}} dy + \underbrace{\int_{|y|>\varepsilon} \varphi_n(y) \underline{\|\tau_y f - f\|_p}}_{\xrightarrow{n\to\infty} 0}$

$(*)$

$\forall \delta > 0$, ① $\exists\, \varepsilon > 0 \ni |y| < \varepsilon \Rightarrow \|\tau_y f - f\|_p < \delta$.

② Given ε, $\exists n_0$ large $\ni \forall n \geq n_0$, $\int_{|y|>\varepsilon} \varphi_n(y)\, dy < \delta$.

$$① + ② + \circledast \Rightarrow \|\varphi_n * f - f\|_p \leq \delta \int_{|y|<\varepsilon} \varphi_n(y)\,dy + \delta\,(2\|f\|_p)$$

$$\leq \delta\,(1 + 2\|f\|_p).$$

QED.

10.3. **Fourier Series.** Let $X = [0,1]$ with the Lebesgue measure. For $n \in \mathbb{Z}$ define $e_n(x) = e^{2\pi i n x}$, and given $f, g \in L^2(X, \mathbb{C})$ define $\langle f, g \rangle = \int_X f \bar{g} \, d\lambda$. This defines an *inner product* on $L^2(X)$, and $\|f\|_{L^2}^2 = \langle f, f \rangle$.

($\bar{g} = g$ Complex conj).

Definition 10.17. If $f \in L^2$, $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f by $\hat{f}(n) = \langle f, e_n \rangle$.

Definition 10.18. For $N \in \mathbb{N}$, let $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n$, be the N-th partial sum of the *Fourier Series* of f.

Question 10.19. *Does $S_N f \to f$? In what sense?*

(Finite dim I.P. space: $\{e_1, \dots, e_N\}$ an $\overline{\text{O.N.}}$ basis ($\langle e_i, e_j \rangle = \delta_{ij}$)

then $\forall v \in V$, $v = \underbrace{\sum_{1}^{N} \langle v, e_i \rangle}\, e_i$

$\hat{f}(n)$ -

Lemma 10.20. $\langle e_n, e_m \rangle = \delta_{n,m}$.

$\forall p \in \mathrm{span}\{e_{-N}, \cdots e_N\}$

Corollary 10.21. *Let $p \in \mathrm{span}\{e_{-N}, \ldots, e_N\}$. Then $\langle f - S_N f, p\rangle = 0$. Consequently, $\|f - S_N f\|_2 \leqslant \|f - p\|_2$.*

Pf: $\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i n x} e^{-2\pi i m x}\, dx = \int_0^1 e^{2\pi i (n-m) x}\, dx = \delta_{m,n}$.

Pf: Note $S_N f = \sum_{-N}^{N} \hat{f}(n) e_n \Rightarrow \langle S_N f, e_m \rangle = \begin{cases} \hat{f}(m) & |m| \le N. \\ 0 & |m| > N. \end{cases}$

$\Rightarrow \forall |m| \le N, \quad \langle S_N f, e_m \rangle = \hat{f}(m) = \langle f, e_m \rangle$.

$\Rightarrow \langle S_N f - f, e_m \rangle = 0 \quad \forall \; |m| \le N$

linearly $\Rightarrow \langle S_N f - f, p \rangle = 0 \quad \forall \; p \in \mathrm{Span}\{e_{-N}, \cdots e_N\}$.

Also NTS. $\|f - S_N f\|_2 \le \|f - p\|_2 \quad \forall p \in \operatorname{span}\{e_{-N}, \dots, e_N\}$.

Pf: Note

$$f - S_N f = (f - p) + \underbrace{(p - S_N f)}_{\in \operatorname{span}\{e_{-N}, \dots, e_N\}}.$$

$$\Rightarrow \langle f - S_N f, p - S_N f \rangle = 0$$

$$\Rightarrow \|f - S_N f\|_2 \le \|f - p\|_2.$$

QED.

Proposition 10.22. $S_N f = D_N * f$, *where* $D_N = \dfrac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(\pi x)}$. *The functions* D_N *are called the* Dirichlet Kernels.

Pf:
$$S_N f(x) = \sum_{-N}^{N} \hat{f}(n)\, e_n(x) = \sum_{-N}^{N} \int_0^1 f(y)\, e^{-2\pi i n y}\, dy \cdot e^{+2\pi i n x}$$

$$= \int_0^1 \Big(\underbrace{\sum_{-N}^{N} e^{2\pi i n(x-y)}}_{D_N(x-y) =} \Big) f(y)\, dy.$$

(sum & get the formula).

$$\Rightarrow S_N f(x) = D_N * f(x)$$

Proposition 10.23. *Define the* Cesàro sum *by* $\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_n f$. *Then* $\sigma_N f = F_N * f$, *where* $F_N = \frac{1}{N}\left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2$.

Remark 10.24. The functions F_N are called the *Fejér Kernels.*

Proposition 10.25. *The Fejér kernels are an approximate identity, but the Dirichlet kernels are not.*

$$\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_n f = \frac{1}{N}\sum_0^{N-1} D_n * f = \underbrace{\left(\frac{1}{N}\sum_0^{N-1} D_n\right)}_{\text{Call this } F_N} * f$$

Have formula can check explicitly.

(Have formulae & can sum & check $F_N =$)

Corollary 10.26. *If $p \in [1, \infty)$ and $f \in L^p$, then $\sigma_N f \to f$ in L^p.*

Corollary 10.27. *If $f \in L^2$ then $S_N f \to f$ in L^2.*

Remark 10.28. If $f \in L^p$ for $p \neq 2$ we need not have $S_N f \to f$ in L^p.

Pf: Note $\|S_N f - f\|_2 \leq \|\sigma_N f - f\|_2$ (only for $p = 2$)

($\because \sigma_N f \in \operatorname{span}\{e_{-N}, \dots, e_N\}$)

$\|\sigma_N f - f\|_p \longrightarrow 0 \ \forall p \in [1, \infty) \Rightarrow$ QED.