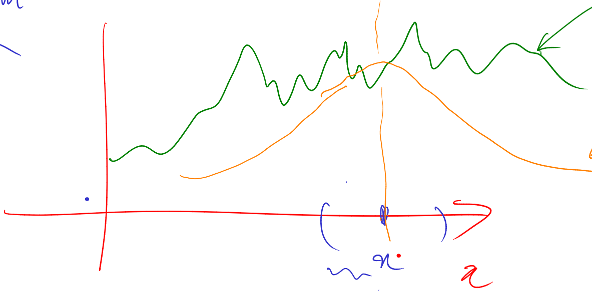
10.2. Convolutions.

Definition 10.8. If $f, g \in \boxed{L^1(\mathbb{R}^d)}$ define the *convolution* by $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$.

Remark 10.9. If $f, g \in L^1(\mathbb{R}^d)$, then $\boxed{f * g} < \infty$ almost everywhere.

Smoothed version

Rough fn f.



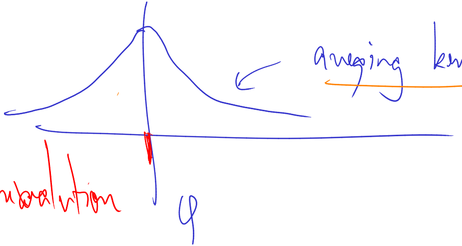
φ translated

$$F(x) = \int_{\mathbb{R}} f(y)\,\varphi(y-x)\,dy$$

negate this (helps later)

$$\boxed{F(x) = \int_{\mathbb{R}} f(y)\,\varphi(x-y)\,dy}$$

Convolution



averaging kernel $(\varphi \geq 0, \int \varphi = 1)$

Note: $\displaystyle\int_{x\in\mathbb{R}^d}\left(\int_{y\in\mathbb{R}^d}|f(y)|\,|g(x-y)|\,dy\right)dx$

(Tonelli) $\displaystyle= \int_{y\in\mathbb{R}^d}\int_{x\in\mathbb{R}^d}|f(y)|\,|g(x-y)|\,dx\,dy$

$\displaystyle= \int_{y\in\mathbb{R}^d}|f(y)|\,\|g\|_1\,dy \le \|f\|_1\,\|g\|_1$

$\Rightarrow |f*g| < \infty$ a.e. (In fact $\|f*g\|_1 \le \|f\|_1\,\|g\|_1$)

Theorem 10.10 (Young). *If* $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$, $f\in L^p(\mathbb{R}^d)$, $g\in L^q(\mathbb{R}^d)$ *then* $f*g\in L^r(\mathbb{R}^d)$, *and* $\|f*g\|_{L^r}\leqslant\|f\|_{L^p}\|g\|_{L^q}$.

Remark 10.11. One can show $\|f*g\|_r\leqslant C_{p,q}\|f\|_p\|g\|_q$ for some constant $C_{p,q}<1$. The optimal constant can be found by choosing f,g to be Gaussian's.

Pf:

Dimension count: $\|f\|_p \sim L^{d/p}$ $\quad f*g \sim L^d$

$\|g\|_q \sim L^{d/q}$ $\quad \|f*g\|_r \sim L^{d+d/r}$

Equating dimensions $\frac{d}{p}+\frac{d}{q}=d+\frac{d}{r}$

① Use duality. WLOG $f,g\geq 0$. Let $h\in L^{r'}$, $h\geq 0$

Let p', q', r' be the Holder conj of p,q,r.

$\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{r}+\frac{1}{r'}=1$

(2) $\int_{\mathbb{R}^d} f*g(x)\, h(x)\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\, g(y)\, h(x)\, dy\, dx$

(*) $$= \int_{\mathbb{R}^{2d}} \Big(\underbrace{f(x-y)^{\frac{p}{r}}\, g(y)^{\frac{q}{r}}}_{L^{r}} \cdot \underbrace{f(x-y)^{\frac{p}{q'}}\, h(x)^{\frac{r'}{q'}}}_{L^{q'}} \ \underbrace{g(y)^{\frac{q}{p'}}\, h(x)^{\frac{r'}{p'}}}_{L^{p'}} \Big)\, dx\, dy$$

$$\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r} \Rightarrow 1-\frac{1}{p'} + 1-\frac{1}{q'} = 1+\frac{1}{r} \Leftrightarrow \frac{1}{p'}+\frac{1}{q'}+\frac{1}{r} = 1$$

$$\Rightarrow \frac{1}{q'}+\frac{1}{r} = \frac{1}{p} \Rightarrow \text{(1)}\ \boxed{\frac{p}{q'}+\frac{p}{r}=1} \qquad \text{(2)}\ \boxed{\frac{q}{p'}+\frac{q}{r}=1}$$

$$③\quad \frac{q'}{p'} + \frac{r'}{q'} = 1$$

Holder

$$\circledast \le \left[\int_{\mathbb{R}^{2d}} \left(f(x-y)^{p/r}\, g(y)^{q/r} \right)^{r} dx\,dy \right]^{1/r} \cdot \left[\int \left(\quad \right)^{q'} dx\,dy \right]^{\frac{1}{q'}} \left[\int \left(\quad \right)^{p'} \right]^{1/p'}$$

Tonelli

$$\le \|f\|_p^{p/r} \|g\|_q^{q/r} \cdot \|f\|_p^{\frac{p}{q'}} \|h\|_{r'}^{\frac{r'}{q'}} \|g\|_q^{q/p'} \|h\|_{r'}^{r'/p'}$$

$$= \|f\|_p \|g\|_q \|h\|_{r'}$$

Gen Holder:
$$\sum \frac{1}{p_i} = 1 \quad , \quad p_i \in [1,\infty].$$
$$\Rightarrow \left| \int \prod f_i \right| \leq \prod \|f_i\|_{p_i} \qquad (\text{Holder} + \text{ind}).$$

$$\Rightarrow \left| \int (f*g) \cdot h \right| \leq \|f\|_p \|g\|_q \|h\|_{r'}$$

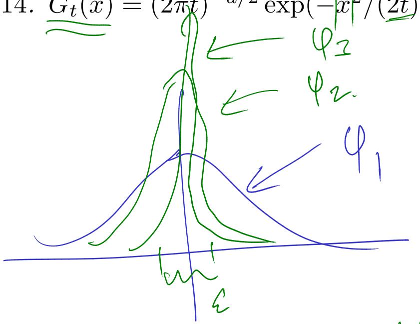
$$\Rightarrow (f*g) \in L^r \;\; \& \;\; \|f*g\|_r \leq \|f\|_p \|g\|_q$$

Q.E.D.

Definition 10.12. $\boxed{(\varphi_n) \text{ is an } \textit{approximate identity}}$ if: (1) $\varphi_n \geqslant 0$, (2) $\boxed{\int_{\mathbb{R}^d} \varphi_n = 1}$, and (3) $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 10.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\boxed{\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})}$. (A.I. as $\varepsilon \to 0$).

Example 10.14. $G_t(x) = (2\pi t)^{-d/2} \exp(-x^2/(2t))$.



$G : \mathbb{R}^d \longrightarrow \mathbb{R}$.

$$G(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$$

$$\forall t > 0, \quad G_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}.$$

Proposition 10.15. *If $p \in [1, \infty)$, $f \in L^p$, and (φ_n) is an approximate identity, then $\varphi_n * f \to f_n$ in L^p.*

Remark 10.16. For $p = \infty$ the above is still true at points where f is continuous.