$X \times Y$: $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$

$f : X \times Y \longrightarrow \mathbb{R}$ $\boxed{\mu \,\&\, \nu \text{ are } \sigma\text{-finite}}$

① $\exists!\ \pi$ on $\Sigma \otimes \tau$ s.t. $\pi(A \times B) = \mu(A)\,\nu(B)$ (Done)

② $\forall f : X \times Y \to \underline{[0,\infty)}$: $\displaystyle\int_{X \times Y} f \, d\pi = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) =$

(Tonelli's)

$$= \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) \quad (*)$$

③ (Fubini) $(*)$ holds if $f \in \mathcal{L}^1(X \times Y, \Sigma \otimes \tau)$.

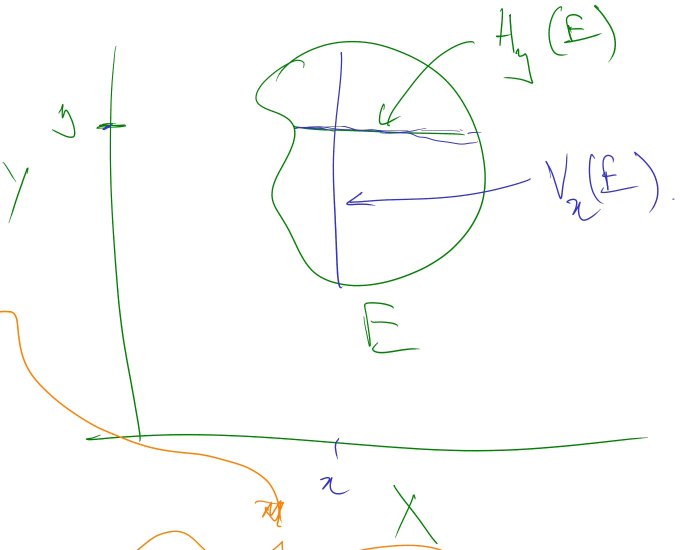
Proof of Theorem 10.2 (Tonelli)

Recall last time

$$\rightarrow \pi(E) \overset{1}{=} \int_{y\in Y} \mu(H_y(E))\, d\nu(y)$$

Today (uniqueness of π)

$$\overset{2}{=} \int_{x\in X} \nu(V_x E)\, d\mu(x)$$



$$\rightarrow \pi(E) = \int_{X\times Y} \mathbb{1}_E \, d\pi \overset{1}{=} \int_{y\in Y} \left(\int_{x\in X} \mathbb{1}_E(x,y)\, d\mu(x) \right) d\nu(y)$$

$$\overset{②}{=} \int_{x \in X} \left(\int_{y \in Y} \mathbb{1}_E(x,y)\, d\nu(y) \right) d\mu(x).$$

$\Rightarrow$ Tonelli is true for indicator fns.

$\Rightarrow$ Tonelli is true for simple fns (linearity)

$\Rightarrow$ Tonelli is true for ~~#~~ +ve fns (Monotone Conv).

QED.

Proof of Theorem 10.3 (Fubini).

$$f \in \mathcal{L}^1(X \times Y) \Rightarrow \int_{X\times Y} |f| \, d\pi \overset{\text{Tonelli}}{=} \int_X \left(\int_Y |f(x,y)| \, d\nu(y) \right) d\mu(x) < \infty$$

$$\Rightarrow \tilde{\forall} x \in X, \int_Y |f(x,y)| \, d\nu(y) < \infty \Rightarrow \tilde{\forall} x \in X, \ f(x,y) \text{ is } \underline{\text{int}} \ \underline{\text{as}} \text{ a } \underline{\text{f}}\text{n } \underline{\text{of}} \ \underline{y}.$$

Also, $\int_{X\times Y} f \, d\pi = \int_{X\times Y} (f^+ - f^-) \, d\pi \overset{\text{Tonelli}}{=} \int_X \left(\int_Y f^+(x,y) \, d\nu(y) \right) d\mu(x)$

$$- \int_X \left(\int_Y f^-(x,y) \, d\nu(y) \right) d\mu(x)$$

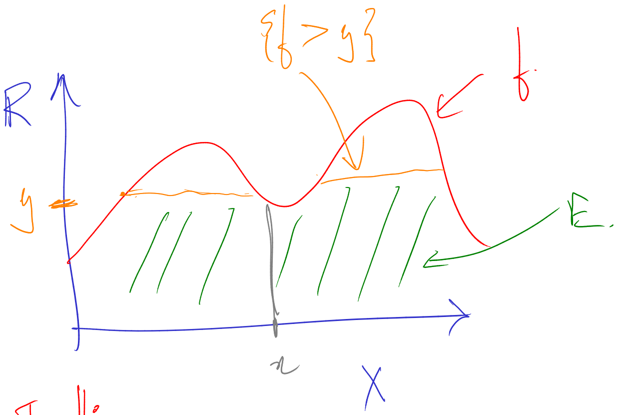
$$= \int_X \left(\int_Y \underbrace{\left(f^+(x,y) - f^-(x,y) \right)}_{f(x,y)} d\nu(y) \right) d\mu(x)$$

$\Rightarrow$ QED Fubini!

Theorem 10.5 (Layer Cake). *If $f : X \to [0,\infty]$ is measurable then* $\displaystyle\int_X f\,d\mu = \int_0^\infty \mu(f > t)\,dt.$

(Assume μ is σ-finite)

Pf: Let π = product measure on $X \times \mathbb{R}$
μ λ



$$E \subset X \times Y = \{(x,y) \mid 0 \le y \le f(x)\}.$$

$$\pi(E) = \int_{x \in X} \left(\int_{y \in \mathbb{R}} \mathbb{1}_E \, d\lambda(y) \right) d\mu(x) \overset{\text{Tonelli}}{=} \int_{y \in \mathbb{R}} \left(\int_{x \in X} \mathbb{1}_E(x,y) \, d\mu(x) \right) dy$$

$$= \int_{x \in X} f(x)\, d\mu \qquad = \int_{y=0}^{\infty} \mu(f > y)\, dy \quad \text{QED.}$$

Proposition 10.6. *If* $(a_{m,n})$ *are such that* $\sum_{m,n=0}^{\infty} |a_{m,n}| < \infty$*, then* $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n}$.

Pf: Fubini using the counting measure.

Theorem 10.7 (Minkowski's inequality). *If $f\colon X \times Y \to \mathbb{R}$ is measurable, then* $p \in [1, \infty]$.

$$\Big(\int_X \Big|\int_Y f(x,y)\,d\nu(y)\Big|^p\,d\mu(x)\Big)^{1/p} \leqslant \int_Y \Big(\int_X |f(x,y)|^p\,d\mu(x)\Big)^{1/p}\,d\nu(y)$$

Pf: Intuition:

$$F(x) = \int_{y \in Y} |f(x,y)|\,d\nu(y).$$

Guess

$$\|F\|_{L^p(X)} \leq \int_{y \in Y} \|f(\cdot, y)\|_{L^p(X)}\,d\nu(y)$$

Pf: Let $g \in L^q(X)$, $\frac{1}{p} + \frac{1}{q} = 1$

Compute $\displaystyle\int_X F(x)\,|g(x)|\,d\mu(x) = \int_X \int_{y\in Y} |f(x,y)|\,d\nu(y)\,|g(x)|\,d\mu(x)$

$$\overset{\text{Tonelli}}{=} \int_{y\in Y} \left(\int_{x\in X} |f(x,y)|\,|g(x)|\,d\mu(x) \right) d\nu(y)$$

$$\overset{\text{Holder}}{\leq} \int_{y\in Y} \left(\|g\|_{L^q(X)} \cdot \|f(\cdot,y)\|_{L^p(X)} \right) d\nu(y)$$

$$\leq \|g\|_{L^q(X)} \cdot \int_{y\in Y} \|f(\cdot,y)\|_{L^p(X)}\,d\nu(y)$$

$\therefore$ By Duality, $\|F\|_{L^p} \leq \int_{y \in Y} \|f(\cdot, y)\|_{L^p(\underline{X})} \, d\nu(y)$

QED