10. Product measures

Rectangles. Product σ-alg.

Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\ B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 10.1. *Let μ, ν be two σ-finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.*

Theorem 10.2 (Tonelli)**.** *Let $f : X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$-measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,*

Iterated Integrals

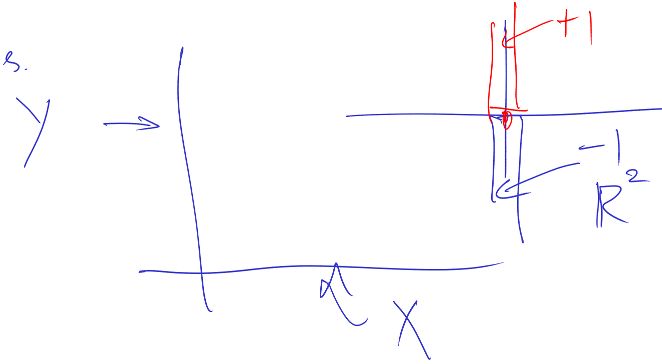
$$\int_{X \times Y} f(x,y)\, d\pi(x,y) = \int_{x \in X} \left(\int_{y \in Y} f(x,y)\, d\nu(y) \right) d\mu(x) = \int_{y \in Y} \left(\int_{x \in X} f(x,y)\, d\mu(x) \right) d\nu(y)\,. \tag{10.1}$$

Theorem 10.3 (Fubini)**.** *If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover,(10.1) holds.*

$f : X \times Y \to \mathbb{R}$. $\Sigma \otimes \tau$ meas.

$\forall\ x_0 \in X$ $\quad y \longmapsto f(x_0, y)$.

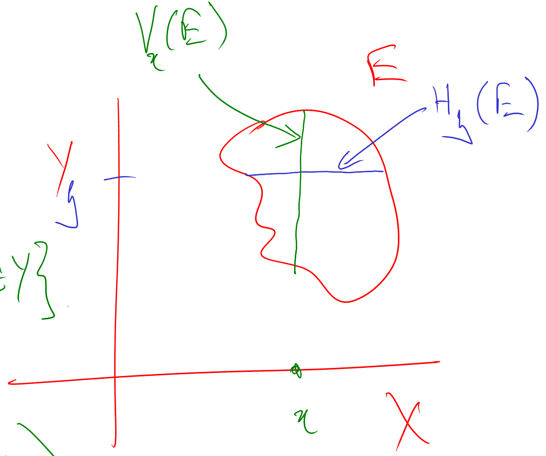
$\forall\ y_0 \in Y$ $\quad x \longmapsto f(x, y_0)$



Lemma 10.4. *For every $E \subseteq X \times Y$, $x \in X$, $y \in Y$ define the horizontal and vertical slices of E by $H_y(E) = \{x \in X \mid (x, y) \in E\}$ and $V_x(E) = \{y \in Y \mid (x, y) \in E\}$.*

(1) *For every $x \in X$, $y \in Y$ we have $H_y(E) \in \Sigma$ and $V_x(E) \in \tau$.*

(2) *The functions $\underbrace{x \mapsto \nu(V_x(E))}_{\Sigma\text{-meas}}$ and $\underbrace{y \mapsto \mu(H_y(E))}_{\tau\text{-meas.}}$ are measurable.*



Pf: (1) $\Lambda = \{E \in \Sigma \otimes \tau \mid H_y(E) \in \Sigma \ \forall y \in Y\}$

Claim: Λ is a σ-alg.

(Pf: $H_y\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} H_y(E_i)$) $\Rightarrow$ QED (1) of lemma.

NTS (2): I.e. NTS. the fn $y \mapsto \mu(H_y(E))$ is τ-meas.

Case I: μ & ν are finite

Pf: $\Lambda = \{ E \in \Sigma \otimes \tau \mid$ the fn $y \longmapsto \mu(H_y(E))$ is τ meas$\}$.

Dynkin Systems: ① $\Lambda \supseteq \Sigma \times \tau$ (rectangles) which is a π-sys.

② $E, F \in \Lambda$, $E \subseteq F$, then $F - E \in \Lambda$

(Pf: $\mu\Big(H_y(F-E)\Big) = \underbrace{\mu\big(H_y(F)\big)}_{\text{meas wrt } y} - \underbrace{\mu\big(H_y(E)\big)}_{\text{meas wrt } y}$ ($\because \mu, \nu$ are finite).

meas wrt y ($\because E, F \in \Lambda$)

$\Rightarrow y \longmapsto \mu(H_y(F-E))$ is τ meas

$$\Rightarrow F - E \in \Lambda.)$$

③ $E_n \in \Lambda$, $E_n \subseteq E_{n+1}$.

$$\mu\left(H_y\left(\bigcup_1^\infty E_n\right)\right) = \lim_{n\to\infty} \underbrace{\mu(H_y(E_n))}_{\tau\text{-meas fns of } y}$$

$$\Rightarrow \underbrace{\qquad\qquad\qquad\qquad}_{\text{is a } \tau \text{ meas fn of } y}$$

$$\Rightarrow \bigcup_1^\infty E_n \in \Lambda.$$

$\therefore \Lambda$ is a λ-sys & $\Lambda \supseteq \Sigma \times \mathcal{T} \Rightarrow \Lambda \supseteq \sigma(\Sigma \times \mathcal{T}) = \Sigma \otimes \mathcal{T}$

QED.

Case II: μ, ν σ-finite: $X = \bigcup F_n$, $Y = \bigcup E_n$.

$\mu(F_n) < \infty$, $\nu(E_n) < \infty$, $F_n \subseteq F_{n+1}$, $E_n \subseteq E_{n+1}$

$$\mu(H_y(A)) = \lim_{n\to\infty} \underbrace{\mu(H_y(A \cap (E_n \times F_n)))}_{\text{by case 1 are all } \tau\text{-meas fns.}}$$

$\Rightarrow y \to \mu(H_y(A))$ is also τ-meas.

QED.

Proof of Theorem 10.1. NTS $\exists!$ meas π $\ni$ $\pi(A \times B) = \mu(A)\,\nu(B)$.

① Uniqueness $\longrightarrow$ Done before

π_1 & π_2 are 2 product measures

Know $\pi_1 = \pi_2$ on $\Sigma \times \tau$ (π-system)

Know μ & ν are σ-finite

$\Bigg\} \Longrightarrow \pi_1 = \pi_2$ on $\sigma(\Sigma \times \tau) = \Sigma \otimes \tau$.

② IOU Existence.

Let $\pi(E) = \displaystyle\int_{y \in Y} \mu\big(H_y(E)\big)\, d\nu(y)$

(integral is defined $\because$ $y \to \mu(H_y(E))$ is τ - meas & ≥ 0)

(a) Is π a measure?

Say $E_n \subseteq \Sigma \otimes \tau$, $E_n \cap E_m = \phi$ if $n \neq m$.

$$\pi\left(\bigcup_1^\infty E_n\right) = \int_{y \in Y} \mu\left(H_y\left(\bigcup_1^\infty E_n\right)\right) \, d\nu(y)$$

$$= \int_{y \in Y} \sum_1^\infty \mu(H_y(E_n)) \, d\nu(y) \overset{MC}{=} \sum_1^\infty \int_{y \in Y} \mu(H_y(E_n)) \, d\nu(y)$$

$$= \sum_{1}^{\infty} \pi(E_n) \quad \Rightarrow \pi \text{ is a meas.}$$

(2) $$\pi(A \times B) = \int_{y \in Y} \mu(H_y(A \times B))\, d\nu(y)$$

$$= \int_{y \in Y} \mathbb{1}_B(y)\, \mu(A) = \nu(B)\, \mu(A) \quad \text{QED.}$$