Theorem 9.25. *Let (X, Σ, μ) be a σ-finite measure space, $p \in [1, \infty)$, $1/p + 1/q = 1$. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.*

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

Pf: $g \in L^q$ $\quad T_g \in (L^p)^*$ $\quad T_g f = \int_X f g \, d\mu$.

$$\|T_g\|_{(L^p)^*} = \sup_{f \in L^p, \|f\|_p = 1} |T_g(f)| = \sup_{f \in L^p, \|f\|_p = 1} \left| \int_X f g \, d\mu \right|$$

Duality Lemma $\quad \|g\|_{L^q}$

$\Rightarrow g \to T_g$ is an isom from $L^q \to (L^p)^*$

NTS $g \to T_g$ is surjective. i.e. If $\Lambda \in (L^p)^*$, NTS $\exists g \in L^q$ s.t. $T_g = \Lambda$

Case I: Say μ is finite.

Define $\nu(A) = \Lambda(\mathbb{1}_A)$.

Claim: ν is a measure!

Pf: Say $A_1, A_2, \dots$ ctably many disj sets

Claim: $\mathbb{1}_{\bigcup_1^N A_n} \xrightarrow{L^p} \mathbb{1}_{\bigcup_1^\infty A_n}$.

Pf: $\| \mathbb{1}_{\bigcup_1^\infty A_n} - \mathbb{1}_{\bigcup_1^N A_n} \|_p^p = \int_X \left(\sum_{N+1}^\infty \mathbb{1}_{A_n} \right)^p \overset{\text{disj}}{=} \sum_{N+1}^\infty \mu(A_n)^* \xrightarrow[N\to\infty]{} 0$

$$\Rightarrow \nu\left(\bigcup_1^\infty A_n\right) = \Lambda\left(\mathbb{1}_{\bigcup_1^\infty A_n}\right) \overset{\text{claim \& continuity of } \Lambda}{=} \lim_{N\to\infty} \Lambda\left(\mathbb{1}_{\bigcup_1^N A_n}\right) \qquad (\because \mu(X) < \infty).$$

$$= \lim_{N\to\infty} \sum_1^N \nu(A_n) \qquad (\text{linearity of } \Lambda)$$

$$= \sum_1^\infty \nu(A_n).$$

$\Rightarrow$ ν is a signed measure

Claim: $\nu \ll \mu$. (Pf: $\mu(A) = 0 \Rightarrow \nu(A) = \Lambda(\mathbb{1}_A) = \Lambda(0) = 0$)

$(\because \mathbb{1}_A = 0$ a.e.$)$

$\Rightarrow$ By R.N. $\exists g$ ~~meas~~ int. $\ni$ $\nu(A) = \int_A g \, d\mu$.

Claim: ① $g \in L^q$. Claim ②: $\Lambda(s) = \int_X s g \, d\mu$ $\forall s$ simple.

Claim ③: $\Lambda(f) = \int_X f g \, d\mu$ $\forall f \in L^p$.

Note Pf of 2: $\Lambda(\mathbb{1}_A) = \int \nu(A) = \int_A g \, d\mu$ & linearity $\Rightarrow$ Claim ② QED.

Pf of Claim ①: Know $\|g\|_{L^q} = \sup_{\|f\|_p} \int_X fg \, d\mu \underset{\text{you check}}{=} \sup_{\substack{s \text{ simple} \\ \|s\|_p = 1}} \int_X sg \, d\mu$

$$= \sup_{\substack{s \text{ simple} \\ \|s\|_p = 1}} \Lambda(s) \quad (\text{by Claim } 2)$$

$$\leq \|\Lambda\|_{(L^p)^*} < \infty \Rightarrow g \in L^q.$$

D.C. $\Rightarrow$ Claim 3 $\Rightarrow$ QED.

Case II: $X = \bigcup F_n$, $F_n \subseteq F_{n+1}$ & $\mu(F_n) < \infty$.

Note the fn g from case I is unique $\quad \Lambda(\mathbb{1}_{F_{n+1}} f) = \int_{F_{n+1}} f g_{n+1}$

$\Rightarrow \forall n \ \exists g_n \in L^q$ + $\Lambda(\mathbb{1}_{F_n} f) = \int_X g_n f \mathbb{1}_{F_n} \, d\mu$.

By uniqueness, $g_{n+1} \mathbb{1}_{F_n} = g_n \mathbb{1}_{F_n}$. Let $g = \lim_{n\to\infty} g_n$ (must exist)

Claim: $g \in L^q$. (Pf: $\int_X |g|^q \underset{MC}{=} \lim \int_{F_n} |g|^q = \lim \int_{F_n} |g_n|^q$

$$= \lim_{n\to\infty} \| \Lambda|_{F_n} \|_{(L^p(F_n))^*} \le \|\Lambda\|_{(L^p(X))^*} \Big)$$

Now $\forall f \in L^p$, $\displaystyle\int_X fg \underset{\substack{\uparrow \\ DC}}{=} \lim_{n\to\infty} \int_{F_n} fg = \lim_{n\to\infty} \Lambda(1_{F_n} f) = \Lambda(f)$

$(\because fg \in L^1)$

QED.

9.4. **Riesz Representation Theorem.**

Theorem 9.28 (Riesz Representation Theorem)**.** *Let X be a compact metric space, and $\mathcal{M}$ be the set of all finite signed measures on X. Define $\Lambda\colon \mathcal{M} \to C(X)^*$ by $\Lambda_\mu(f) = \int_X f\, d\mu$ for all $\mu \in \mathcal{M}$ and $f \in C(X)$. Then Λ is a bijective linear isometry.*

Remark 9.29. In particular, for every $I \in C(X)^*$, there exists a unique finite regular Borel measure μ such that $I(f) = \int_X f\, d\mu$ for every $f \in C(X)$.

μ a finite signed measure on X

$f \in C(X)$. $\qquad T_\mu(f) = \int_X f\, d\mu$

$$|T_\mu(f)| \leq \|f\|_\infty \underbrace{|\mu|(X)}_{\|\mu\|}$$

$$(L^p)^* = \{ \Lambda \mid \Lambda : L^p \longrightarrow \mathbb{R} \text{ is } \underbrace{\text{bdd}}_{\text{cts}} \text{ \& linear} \}$$

$g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $T_g \in (L^p)^*$ def by

$$T_g(f) = \int_X fg \, d\mu$$

10. Product measures

← rectangles

Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\ B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 10.1. *Let μ, ν be two σ-finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.*

Theorem 10.2 (Tonelli)**.** *Let $f \colon X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$-measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,*

$$\int_{X \times Y} f(x,y)\, d\pi(x,y) = \int_{x \in X} \Big(\int_{y \in Y} f(x,y)\, d\nu(y) \Big)\, d\mu(x) = \int_{y \in Y} \Big(\int_{x \in X} f(x,y)\, d\mu(x) \Big)\, d\nu(y)\,. \tag{10.1}$$

Theorem 10.3 (Fubini)**.** *If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover,*(10.1) *holds.*