Theorem 9.16. *Let μ, ν be positive measures such that ν is σ-finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.*

Last time $\longrightarrow$ did the proof of existence when ν is finite.

Uniqueness: Say $\nu = \nu_{ac} + \nu_s = \tilde{\nu}_{ac} + \tilde{\nu}_s$

$$\left.\begin{array}{l} \exists N \ni \mu(N) = 0 \ \& \ \nu_s(N^c) = 0 \\ \exists \tilde{N} \ni \mu(\tilde{N}) = 0 \ \& \ \tilde{\nu}_s(\tilde{N}^c) = 0 \end{array}\right\} \begin{array}{l} \hat{N} = N \cup \tilde{N} \\ (\mu(\hat{N}) = 0 \end{array}$$

Pick any $A \in \Sigma$. $(\nu_s - \tilde{\nu}_s)(A) = (\nu_s - \tilde{\nu}_s)(A \cap \hat{N}) = 0$

also $(\nu_s - \tilde{\nu}_s)(A) = (\nu_{ac} - \tilde{\nu}_{ac})(A) = (\nu_{ac} - \tilde{\nu}_{ac})(\underbrace{A \cap \hat{N}}_{\mu(A \cap \hat{N}) = 0}) = 0$

$$\Rightarrow \quad \nu_s(A) = \tilde{\nu}_s(A) \quad \& \quad \nu_{ac}(A) = \tilde{\nu}_{ac}(A) \quad \Rightarrow \text{uniq QED.}$$

Case II: ν is σ-finite.

Write $X = \bigcup_{1}^{\infty} F_n$, $F_n \subseteq F_{n+1}$ & $\nu(F_n) < \infty$.

$\forall n$, let $\nu^{(n)}(A) = \nu(A \cap F_n)$. Case I $\Rightarrow$ Can write $\nu^{(n)} = \nu^{(n)}_{ac} + \nu^{(n)}_{s}$.

also $\exists\, N_n \subseteq F_n \ni \mu(N_n) = 0$, & $\nu_s^{(n)}(A) = \nu^{(n)}(A \cap N_n)$

$$= \nu(A \cap N_n \cap F_n).$$

Now set $N = \bigcup_{1}^{\infty} N_n$, $\nu_{ac}(A) = \nu(A \cap N^c)$ & $\nu_s(A) = \nu(A \cap N)$.

Clearly $\nu_s \perp \mu$ $\quad (\because \mu(N) = 0)$

NTS $\nu_{ae} \ll \mu$: Say $\mu(A) = 0$. Then $\nu_{ae}(A) = \lim \nu_{ae}^{(n)}(A) = 0$

QED

Corollary 9.17. *Let μ be a positive measure, and ν be a finite* $\boxed{\textit{signed measure.}}$ *There exists a unique pair of signed measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$ and $\nu = \nu_{ac} + \nu_s$.*

Pf: $\nu = \nu^+ - \nu^-$ & write $\nu^{\pm} = \nu^{\pm}_{ac} + \nu^{\pm}_s$

QED.

(Radon Nicodym)

Corollary 9.18. *Let μ, ν be σ-finite positive measures. There exists a unique positive measure ν_s and nonnegative measurable function g such that $\mu \perp \nu_s$ and $d\nu = d\nu_s + g\,d\mu$.*

Pf: Know $\nu = \nu_{ac} + \nu_s$. & by RN know $\exists g \ni d\nu_{ac} = g\,d\mu$.

QED.

9.3. Dual of L^p.

Proposition 9.19. *Let U, V be Banach spaces, and $T\colon U \to V$ be linear. Then T is continuous if and only if there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.*

bdd.

Pf: Say T is cts. $\Rightarrow$ T is cts at 0 $\Rightarrow$ $\forall \varepsilon > 0\ \exists \delta > 0 \ni \|u - 0\|_U < \delta \Rightarrow \|Tu - T0\|_V < \varepsilon$

$\Rightarrow \|u\|_U < \delta \Rightarrow \|Tu\|_V < \varepsilon.$

$\Rightarrow \forall u \in U,\quad \left\|\frac{\delta u}{2\|u\|}\right\| = \frac{\delta}{2}. \Rightarrow \left\|T\left(\frac{\delta u}{2\|u\|}\right)\right\| < \varepsilon$

$\Rightarrow$ (linearity) $\|Tu\| \leq \frac{2\varepsilon\|u\|}{\delta}$

Chose $c = \frac{2\varepsilon}{\delta}$ $\Rightarrow$ QED.

Conversely: Assume $\|Tu\|_V \leq c\|u\|_U$. NTS T is cts.

Note $\|Tu_1 - Tu_2\|_V \overset{\text{linear}}{=} \|T(u_1 - u_2)\| \leq c\|u_1 - u_2\|_V$.

$\Rightarrow$ T is Lipschitz $\Rightarrow$ cts

QED

Definition 9.20. We say $T\colon U \to V$ is a $\boxed{\text{bounded linear transformation}}$ if T is linear and there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.

$\leftarrow$ u^* is called a bdd linear functional

Definition 9.21. The *dual* of U is defined by $U^* = \{u^* \mid u^*\colon U \to \mathbb{R}$ is $\boxed{\text{bounded}}$ and linear.$\}$ Define a norm on U^* by

$$\|u^*\|_{U^*} \overset{\text{def}}{=} \sup_{u\in U-0} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U=1} \cancel{\frac{1}{\|u\|_U}} u^*(u) = \sup_{\|u\|_U=1} \cancel{\frac{1}{\|u\|_U}} |u^*(u)|\,.$$

Proposition 9.22. *The dual of a Banach space is a Banach space.*

$\mathcal{L}(U,V) = \{ T \mid T: U \to V \}$ T bdd linear $\}$

$$\|T\|_{\mathcal{L}(U,V)} = \sup_{u \neq 0} \frac{\|Tu\|_V}{\|u\|_U}$$

$\longrightarrow$ (You check)

Proposition 9.23. *Let $1/p + 1/q = 1$, $g \in L^q(X)$. Define $T_g : L^p \to \mathbb{R}$ by $T_g f = \int_X fg \, d\mu$. Then $T_g \in (L^p)^*$.*

$p \in [1, \infty]$

Proposition 9.24. *The map $g \mapsto T_g$ is a bounded linear map from $L^q \to (L^p)^*$.*

$$T_{(g_1+g_2)} f = \int (g_1+g_2) f = T_{g_1} f + T_{g_2} f$$

Pf: Clearly $T_g(f_1 + f_2) = T_g f_1 + T_g f_2$.

also $|T_g f| = \left| \int_X fg \, d\mu \right| \overset{\text{Holder}}{\leq} \|f\|_p \|g\|_q \Rightarrow$ cts.

$\Rightarrow T_g \in (L^p(X))^*$.

ℓ

Also note $\|T_g\|_{(L^p)^*} \leq \|g\|_q$ (Claim: Duality from before $\Rightarrow$ $\|T_g\|_{(L^p)^*} = \|g\|_q$).

Theorem 9.25. *Let (X, Σ, μ) be a σ-finite measure space, $p \in [1, \infty)$, $1/p + 1/q = 1$. The map $g \mapsto T_g$ is a bijective linear isometry between L^q and $(L^p)^*$.*

Remark 9.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 9.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

requires work.
(Next time.)

Pr: $(\Omega, \mathcal{F}, P)$

$P(\Omega) = 1.$

R.V. $\rightarrow$ $X : \Omega \rightarrow \mathbb{R}$ $\mathcal{F}$-meas is a R.V.

RV, Y $\leftarrow$ Observe Y. $\rightarrow$ What events can you decide the tr of?

$\underline{\sigma(Y)} = \sigma$ alg gen by $\{ \underline{Y^{-1}(U)} \mid U \subseteq \mathbb{R} \text{ open} \}$

$$E(X \mid \underline{Y}) = E(\underbrace{X \mid}_{\text{on}} \underbrace{\sigma(Y)}_{\sim})$$

$$\sigma\left(Y^{-1}(U) \mid U \subseteq \mathbb{R} \text{ is open}\right)$$

~~Ex~~ $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(A \mid \mathcal{F}) = E(\mathbb{1}_A \mid \mathcal{F})$$