9.2. Absolute Continuity.

Definition 9.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g\, d\mu$. (Notation: Say $d\nu = g\, d\mu$.)

μ, ν

unique

Theorem 9.15 (Radon-Nikodym). *If μ, ν are two positive measures such that ν is σ-finite and $\nu \ll \mu$, then there exists a measurable function g such that $0 \leqslant g < \infty$ almost everywhere and $d\nu = g\, d\mu$.*

Last time: (1) Case I: $\nu(X) < \infty$. ($\&$ $\mu(X) < \infty$)

(2) $\mathcal{F} = \{ f \mid f \geq 0 \ \& \ \forall A, \int_A f\, d\mu \leq \nu(A) \}$

(3) Note if $f_1, f_2 \in \mathcal{F} \Rightarrow f_1 \vee f_2 \in \mathcal{F}$.

(4) $\Rightarrow \exists$ a seq (f_n) $\ni$ $\int_X f_n\, d\mu \longrightarrow \sup_{f \in \mathcal{F}} \int_X f\, d\mu$ (Exercise: $< \nu(X) < \infty$)

by ③ can ensure $f_n \leq f_{n+1}$

⑤ Let $g = \lim_{n\to\infty} f_n$.

$\boxed{\underline{\text{Claim}} \quad d\nu = g\, d\mu}$

⑥ Let $d\lambda = d\nu - g\, d\mu$ (i.e. $\lambda(A) = \nu(A) - \int_A g\, d\mu$.)

Note: $g \in \mathcal{F}$ (M.C.) $\Rightarrow$ λ is a +ve meas.

⑦ NTS $\lambda = 0$. Will show $\forall \varepsilon > 0$, $\lambda \leq \varepsilon\mu$. ($\Rightarrow$ <u>QED</u>)

Note $\lambda - \varepsilon\mu$ is a signed measure. Let $X = P \cup N$ be the Hahn decomposition of $\lambda - \varepsilon\mu$.

Claim: $g + \varepsilon \mathbb{1}_P \in \mathcal{F}$.

$\hookrightarrow$ Pf: NTS $\forall A$, $\int_A (g + \varepsilon \mathbb{1}_P)\, d\mu \le \nu(A)$

$$\int_A (g + \varepsilon \mathbb{1}_P)\, d\mu = \nu(A) - \lambda(A) + \varepsilon \mu(A \cap P)$$

$$= \nu(A) - \underbrace{\lambda(A \cap N)}_{\ge 0} - \underbrace{\left(\lambda(A \cap P) - \varepsilon \mu(A \cap P)\right)}_{\ge 0 \ (\because P \text{ is } +\text{ve for } \lambda - \varepsilon\mu)}$$

$$\le \nu(A)$$

$\Rightarrow$ Claim.

$\Rightarrow$ $g + \varepsilon \mathbb{1}_P \in \mathcal{F} \Rightarrow \mu(P) = 0 \overset{a.c.}{\Longrightarrow} \nu(P) = 0$

$\int_X g\, d\mu = \sup_{f \in \mathcal{F}} \int_X f\, d\mu$

$$\Rightarrow \lambda(P) = 0 \Rightarrow (\lambda - \varepsilon\mu)(P) = 0$$

$\Rightarrow \lambda - \varepsilon\mu$ is a $-$ve meas

$\Rightarrow \lambda \le \varepsilon\mu. \Rightarrow$ QED.

Uniqueness: If $d\nu = g\,d\mu = h\,d\mu \Rightarrow g = h$ a.e.

Pf: $\forall A$, $\int_A g\,d\mu = \int_A h\,d\mu \Rightarrow \int_A (g-h)\,d\mu = 0 \ \forall A$.

Chose $A = \{g - h > 0\}. \Rightarrow \int_{\{g>h\}} (g-h)\,d\mu = 0 \Rightarrow \mu\{g > h\} = 0$

III^{ly} $\mu\{g < h\} = 0 \Rightarrow g = h$ a.e.

Case II: Write $X = \bigcup F_n$, $\mu(F_n) < \infty$, $\nu(F_n) < \infty$.

WLOG. assume $F_n \subseteq F_{n+1}$.

By Case I, $\exists\, g_n$ + $\forall A$, $\nu(A \cap F_n) = \int_{A \cap F_n} g_n \, d\mu$

By uniqueness $g_{n+1}\big|_{F_n} = g_n$.

Set $g = \lim g_n$ (is an inc lim).

$\nu(A) = \lim_{n \to \infty} \nu(A \cap F_n) = \lim_{n \to \infty} \int_{A \cap F_n} g \, d\mu = \lim_{n \to \infty} \int_A \mathbb{1}_{F_n} g \, d\mu$

$\overset{MC}{=} \int_A g \, d\mu.$ QED.

Theorem 9.16. *Let μ, ν be positive measures such that ν is σ-finite. There exists a unique pair of measures (ν_{ac}, ν_s) such that $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, and $\nu = \nu_{ac} + \nu_s$.*

Pf: Case I : $\cancel{\mu}, \nu$ finite

Let $\mathcal{N} = \{ A \mid \mu(A) = 0 \}$ &

Consider $\sup \{ \nu(A) \mid A \in \mathcal{N} \}$, & find N_k s.t. $\nu(N_k) \xrightarrow{k \to \infty} \sup\limits_{A \in \mathcal{N}} \nu(A)$

Let $N = \bigcup\limits_{k=1}^{\infty} N_k$. & $\nu_s(A) = \nu(A \cap N)$, $\nu_{ac}(A) = \nu(A \cap N^c)$.

Claim :

1. $\nu_s \perp \mu$.
2. $\nu_{ac} \ll \mu$.

Pf of ① : $\nu_s(N^c) = 0$ & $\mu(N) = 0 \Rightarrow$ ①.

Pf of ② : NTS $\mu(A) = 0 \Rightarrow \nu_{ac}(A) = 0$

i.e. NTS $\mu(A) = 0 \Rightarrow \nu(A \cap N^c) = 0$

$$\nu(N) \le \nu(A \cup N) \le \nu(N)$$ $\left(\because \nu(N) = \sup_{\mu(B)=0} \nu(B) \right.$ & $\left. \mu(A \cup N) = 0 \right)$

$(\because \nu \text{ is finite}) \Rightarrow \nu(A - N) = 0 \Rightarrow \nu(A \cap N^c) = 0$

QED.