9. **Signed Measures**

9.1. **Hanh and Jordan Decomposition Theorems.**

Definition 9.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
(2) $\mu(\emptyset) = 0$
(3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$.

Example 9.2. Let $f \in L^1(X, \mu)$, and define ν by $\nu(A) = \int_A f \, d\mu$. Then ν is a signed measure, and we write $d\nu = f \, d\mu$.

Example 9.3. If μ, ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is finite.

Theorem 9.4 (Jordan Decomposition)**.** *Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.*

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all measurable sets $B \subseteq A$.

~~Proposition~~ 9.6. *If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and $\mu(B) \leqslant \mu(A)$.*

Lemma

(Last time)

Theorem 9.7 (Hanh decomposition)**.** *If μ is a signed measure on X, then $X = P \cup N$ where P is positive and N is negative.*

Remark 9.8. The decomposition is unique up to null sets.

Pf: $X = P' \cup N' = P \cup N$ $\quad$ (P, P' +ve $\quad N', N$ -ve)

$$\Rightarrow P = \underbrace{(P \cap P')}_{+\text{ve}} \cup \underbrace{(P \cap N')}_{\text{both +ve \& -ve}}$$

$\Rightarrow$ all subsets of $P \cap N'$ are meas 0

$\Rightarrow P = P' \cup$ null set.

Pf of Existence: ① WL assume $-\infty \notin$ range(μ).

② Let $\alpha = \inf \{\mu(E) \mid E \subseteq X\}$. $\quad$ (α could be $-\infty$)

③ Chose $(\alpha_n) \ni (\alpha_n) \longrightarrow \alpha$ & $\alpha < \alpha_{n+1} < \alpha_n$.

$\Rightarrow \forall n, \ \exists A_n \ni \alpha \leq \mu(A_n) < \alpha_n$

$\Rightarrow$ By lemma $\exists B_n$ negative $\ni$ $B_n \subseteq A_n$.

& $\mu(B_n) \leq \mu(A_n)$

$\Rightarrow \alpha \leq \mu(B_n) < \alpha_n$

④ Let $N = \bigcup_{1}^{\infty} B_n$. Clearly $\alpha \leq \mu(N) < \alpha_n \ \forall n$

& N is $-$ve

$\Rightarrow \mu(N) = \alpha \quad (\Rightarrow \alpha > -\infty)$

5 NTS $P = N^c$ is +ve.

Let $E \subseteq P$. NTS $\mu(E) \geq 0$.

If $\mu(E) < 0 \Rightarrow \mu(E \cup N) = \mu(E) + \mu(N) < \alpha$

($\because \alpha$ is finite)

Contradiction

$\Rightarrow \mu(E) \geq 0 \Rightarrow P$ is +ve $\Rightarrow$ QED.

Definition 9.9. We say two positive measures μ, ν are *mutually singular* if there exists $C \subseteq X$ such that for every $A \in \Sigma$ we have $\mu(A \cap C) = \nu(A \cap C^c) = 0$.

$\mu \perp \nu$

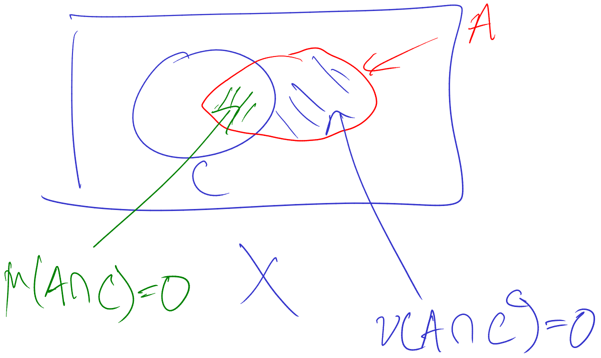
Proof of Theorem 9.4

If μ is a signed meas

then $\exists!\ \mu^+ \ \&\ \mu^- \ni \mu^+ \perp \mu^-$

$\ni \mu = \mu^+ - \mu^-$

(μ^+ & μ^- are +ve meas)



Pf: $X = P \cup N$ by Hahn. Set $\mu^+(A) = \mu(A \cap P)$, $\mu^-(A) = -\mu(A \cap N)$. $\Big\}\Rightarrow$ existence.

Uniqueness $\rightarrow$ $\mu = \mu^+ - \mu^- = \nu^+ - \nu^-$, $\mu^\pm, \nu^\pm \geq 0$, $\mu^+ \perp \mu^-$

$\nu^+ \perp \nu^-$

$$\mu^+ \perp \mu^- \Rightarrow \exists\, C \ni \mu^+(C) = 0 = \mu^-(C^c)$$

$$\nu^+ \perp \nu^- \Rightarrow \exists\, D \ni \nu^+(D) = 0 = \nu^-(D^c).$$

$$\Rightarrow X = \underbrace{C}_{\substack{-\text{ve} \\ \text{wrt } \mu}} \cup \underbrace{C^c}_{\substack{+\text{ve} \\ \text{wrt } \mu}} = \underbrace{D}_{\substack{-\text{ve} \\ \text{wrt } \mu}} \cup \underbrace{D^c}_{\substack{+\text{ve} \\ \text{wrt } \mu.}}$$

Uniqueness of Hahn $\Rightarrow$ QED.

Definition 9.10. Let μ be a signed measure with Jordan decomposition $\boxed{\mu = \mu^+ - \mu^-}$. Define the variation of μ to be the (positive) measure $|\mu| \stackrel{\text{def}}{=} \mu^+ + \mu^-$.

Definition 9.11. Define the *total variation* of μ by $\|\mu\| = |\mu|(X)$. $\in [0, \infty]$

Proposition 9.12. *Let $\mathcal{M}$ be the set of all finite signed measures on X. Then $\mathcal{M}$ is a Banach space under the total variation norm.*

WTS ① $\|\mu + \nu\| \le \|\mu\| + \|\nu\|$ } $\leftarrow$ std def done.

& ② Completeness

Q: $x \in \mathbb{R}$. $\delta_x = \delta$ mass at x: $\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

Q: Does $\delta_{1/n} \longrightarrow \delta_0$ in $\mathcal{M}$? (No: $\|\delta_{1/n} - \delta_0\| = 2$)

Wed

9.2. Absolute Continuity.

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Definition 9.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 9.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g\, d\mu$. (Notation: Say $d\nu = g\, d\mu$.) $\leftarrow$ $\boxed{\int f\, d\nu = \int f g\, d\mu}$

Theorem 9.15 (Radon-Nikodym). *If μ, ν are two σ-finite positive measures with $\nu \ll \mu$, then there exists a measurable function g such that $0 \leqslant g < \infty$ almost everywhere and $d\nu = g\, d\mu$.* $\rightarrow$ $\nu(A) = \int_A g\, d\mu$

Pf:

Case I: μ & ν are finite.

$$\text{Let } \mathcal{F} = \left\{ f \geqslant 0 \;\middle|\; \forall A \in \Sigma,\ \int_A f\, d\mu \leqslant \nu(A) \right\}$$

Q: $\mathcal{F} = \emptyset$? (No $\to f = 0 \in \mathcal{F}$)

Guess: g = "largest element of $\mathcal{F}$" ~~$g(x) = \sup_{f \in \mathcal{F}} f(x)$~~ $\leftarrow$ Won't EVER Work.

Let $\alpha = \sup\limits_{f \in \mathcal{F}} \int_X f \, d\mu \leq \nu(X) < \infty$.

$\Rightarrow \forall n \; \exists f_n \in \mathcal{F} \ni \int_X f_n \, d\mu \geq \alpha - \frac{1}{n}$

(Replace f_n with $\max\{f_n, f_{n-1}\}$ & assume WLOG (f_n) is inc)

Note: $f_1, f_2 \in \mathcal{F} \Rightarrow f_1 \vee f_2 \in \mathcal{F}$ $\quad (a \vee b = \max(a,b))$

$$\left(\because \int_A (f_1 \vee f_2) \, d\mu = \int_{A \cap \{f_1 > f_2\}} f_1 \, d\mu + \int_{A \cap \{f_1 \leq f_2\}} f_2 \, d\mu \leq \nu(A \cap \{f_1 > f_2\}) + \nu(A \cap \{f_1 \leq f_2\}) = \nu(A) \text{ QED} \right).$$

$\#w \Rightarrow \int_X f_n \, d\mu \geq \alpha - \frac{1}{n}$ & $f_{n+1} \geq f_n$ & $f_n \in \mathcal{F}$.

Set $g = \lim f_n$ (must exist).

Next time: $\int_A g \, d\mu = \cancel{\int_A} \nu(A) \quad \forall A \in \Sigma$.