Theorem 8.33 (Vitali). *Let $(f_n) \in L^1(X)$. The sequence (f_n) is convergent in L^1 if and only if*

(1) (f_n) *converges in measure,*

(2) (f_n) *is uniformly integrable,*

(3) (tightness) *there exists $F \in \Sigma$ with $\boxed{\mu(F) < \infty}$ such that $\int_{F^c} |f_n|\, d\mu < \varepsilon$ for all n.*

$\forall \varepsilon > 0,$

Proof:

$\{f_n\}$ is U.I. if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon$ $\forall n$.

Theorem 8.34. *If* $\lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n|\,d\mu = 0$*, then* (f_n) *is uniformly integrable.*

Theorem 8.35. *If there exists an increasing function* $\varphi\colon [0,\infty) \to [0,\infty)$ *such that* $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$*, and* $\sup_n \int_X \varphi(|f_n|)\,d\mu < \infty$*, then* (f_n) *is uniformly integrable.*

superlinear

Remark 8.36. The hypothesis in both the above theorems are equivalent.

Remark 8.37. If additionally $\boxed{\sup_n \int_X |f_n|\,d\mu < \infty}$, then the converse of both the above theorems are true.

Rmk: $f_n \in L^1 \Rightarrow \lim_{\lambda\to\infty} \int_{\{|f_n|>\lambda\}} |f_n|\,d\mu \overset{DCT}{=} 0$

Pf: Pick $\varepsilon > 0$. NTF $\delta > 0 \ni \mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon \quad \forall n$

$\exists \lambda_0 \ni \forall \lambda > \lambda_0, \int_{\{|f_n|>\lambda\}} |f_n| < \varepsilon/2$ (by assumption).

Now $\mu(E) < \delta \Rightarrow \int_E |f_n| = \underbrace{\int_{E \cap \{|f_n| > \lambda\}} |f_n|} + \underbrace{\int_{E \cap \{|f_n| \le \lambda\}} |f_n|}$

$$\le \frac{\varepsilon}{2} + \delta \cdot \lambda$$

Chose $\delta = \frac{\varepsilon}{2\lambda} \Rightarrow$ QED.

Pf of 8.35: $\frac{\varphi(x)}{x} \xrightarrow{x\to\infty} \infty$, φ inc, & $\sup_n \int_X \varphi(|f_n|)\,d\mu < \infty$

NTS $\{f_n\}$ is U.I.

Will show $\lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n|\,d\mu = 0 \Rightarrow$ QED.

Let $\varepsilon > 0$: $\Rightarrow \exists \lambda_0 \ni \underbrace{\forall \lambda > \lambda_0}$ $\frac{\varphi(\lambda)}{\lambda} \geq \frac{1}{\varepsilon} \Leftrightarrow \underline{\lambda \leq \varepsilon\varphi(\lambda)}$.

$$\Rightarrow \int_{\{|f_n|>\lambda_0\}} |f_n| \leq \varepsilon \int_{\{|f_n|>\lambda_0\}} \varphi(|f_n|) \leq \varepsilon \int_X \varphi(|f_n|) \leq \varepsilon \underbrace{\sup_n \int_X \varphi(|f_n|)}$$

Proof:

Proof

$$\Rightarrow \lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n| = 0 \Rightarrow QED.$$

Corollary 8.38. *If $(f_n) \to f$ in measure, $\boxed{\mu(X) < \infty}$, and $\sup_n \|f\|_p < \infty$ for any $p > 1$, then $(f_n) \to f$ in L^q for every $\boxed{q \in [1, p)}$.*

Quick check for $q = 1$:

NTS $(f_n) \longrightarrow f$ in L^1.

Vitali: ETS (f_n) is U.I. (have $(f_n) \xrightarrow{\mu} f$ & tightness)

$\varphi(x) = x^p \quad (p > 1)$

$\sup_n \int_X \varphi(|f_n|) = \sup_n \int_X |f_n|^p < \infty \Rightarrow (f_n)$ is U.I. $\Rightarrow$ (Q.E.D).

9. Signed Measures

(X, Σ) alg on X.

9.1. Hanh and Jordan Decomposition Theorems.

Definition 9.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
(2) $\mu(\emptyset) = 0$
(3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$.

(Claim: If $\sum_1^\infty \mu(A_i)$ is cgt it must be absolutely cgt)

Example 9.2. Let $f \in L^1(X, \mu)$, and define ν by $\nu(A) = \int_A f\, d\mu$. Then ν is a signed measure, and we write $d\nu = f\, d\mu$.

Example 9.3. If μ, ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is ~~finite~~. a signed measure.

Theorem 9.4 (Jordan Decomposition)**.** *Any signed measure can be written (uniquely) as the difference of two mutually singular positive measures.*

IOU.

(Note: μ is a signed measure & $A \subseteq B \not\Rightarrow \mu(A) \leq \mu(B)$.)

Definition 9.5. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all measurable sets $B \subseteq A$.

Proposition 9.6. *If $\mu(A) \in (-\infty, \infty)$ then there exists $B \subseteq A$ such that B is negative and* $\boxed{\mu(B) \leqslant \mu(A)}$.

Lemma ;

Pf: Case ① : $\mu(A) \geq 0$ $\longrightarrow$ Chose $B = \phi$. $\Rightarrow$ QED.

Case ② $\mu(A) < 0$.

If $\sup\{\mu(E) \mid E \subseteq A\} \leq 0 \Rightarrow A$ is $-ve$, chose $B = A \Rightarrow$ QED.

If $\sup\{\mu(E) \mid E \subseteq A\} = \delta_1 > 0$ find & $E_1 \ni \mu(E_1) \geq \frac{\delta_1}{2} \wedge 1$.

② Let $\delta_2 = \sup\{\mu(E) \mid E \subseteq A - E_1\}$ & find $E_2 \ni \mu(E_2) \geq \frac{\delta_2}{2} \wedge 1$

⋮

ⓝ Let $\delta_n = \sup\{\mu(E) \mid E \subseteq A - \bigcup_1^{n-1} E_k\}$ & find $E_n \ni \mu(E_n) \geq \frac{\delta_n}{2} \wedge 1$

If at any stage $\delta_n \le 0 \longrightarrow$ done: $A - \bigcup_1^{n-1} E_k$ is $-$ve $\Rightarrow$ QED.

<u>Claim</u>: $\sum_1^\infty \delta_i < \infty$. $\left(\because \cancel{\mu(A) = \mu\Big(\underbrace{A - \bigcup_1^\infty E_k}_{B} \cup \bigcup_1^\infty E_k\Big)}\right)$

Let $B = A - \bigcup_1^\infty E_k$. Then $\underline{\mu(A)} = \mu\left(B \cup \bigcup_1^\infty E_k\right) = \underline{\mu(B)} + \underbrace{\sum_1^\infty \mu(E_k)}_{\ge 0}$

$\because \mu(A) < \infty \Rightarrow \sum_1^\infty \mu(E_k) < \infty$.

<u>Claim 2</u>: B is $-$ve & $\mu(B) \le \mu(A)$ $(\Rightarrow$ QED$)$.

NTS B is $-$ve.

Note $\sum \mu(E_{n_k}) < \infty \Rightarrow \underline{\sum \delta_k < \infty}$.

Also, $E \subseteq B \Rightarrow E \subseteq A - \bigcup_1^{n-1} E_k \Rightarrow \mu(E) \leq \delta_n \xrightarrow{n\to\infty} 0$

$\Rightarrow \mu(E) \leq 0$.

QED.