Proposition 8.28. *If $p \in [1,\infty)$, $(f_n) \to f$ in L^p, then $(f_n) \to f$ in measure.*

Lemma 8.29 (Chebychev's inequality)**.** *For any $\lambda > 0$, we have $\mu(\{|f| > \lambda\}) \leqslant \frac{1}{\lambda}\|f\|_1$*

Pf: $\int_X \lambda \mathbb{1}_{\{|f|>\lambda\}} d\mu \leq \int_X |f| \mathbb{1}_{\{|f|>\lambda\}} d\mu \leq \|f\|_{L^1}$

$\| \quad \lambda \mu\{|f|>\lambda\}$

$\Rightarrow \mu(|f|>\lambda) \leq \frac{1}{\lambda}\|f\|_1$ QED.

Cor: $\forall p \geq 1$, $\mu\{|f|>\lambda\} \leq \frac{1}{\lambda^p}\|f\|_p^p$ (Pf: $\{|f|>\lambda\} = \{|f|^p > \lambda^p\}$ & chebychev)

Proof of Proposition 8.28

If $(f_n) \longrightarrow f$ in L^p $\Rightarrow$ $\|f_n - f\|_p \xrightarrow{n \to \infty} 0$

$\Rightarrow \forall \varepsilon > 0$ $\mu\{|f_n - f| > \varepsilon\} \leq \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \xrightarrow{n \to \infty} 0$

QED.

Converse? $(f_n) \to f$ in meas does $(f_n) \to f$ in L^p

No: $f_n = \frac{1}{n} \mathbb{1}_{[0, n^2]}$ $\longrightarrow 0$ in meas

$\not\longrightarrow 0$ in L^1

8.3. Uniform integrability.

Question 8.30. *When does convergence in measure imply L^1 convergence?*

Definition 8.31. We say $\{f_\alpha \mid \alpha \in \mathcal{A}\}$ is uniformly integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$ we have $\int_E |f| \, d\mu < \varepsilon$.

$\boxed{\textbf{Proposition 8.32.}}$ *If $|f_\alpha| \leqslant F$ for all $\alpha \in \mathcal{A}$, and $F \in L^1$, then $\{f_\alpha \mid \alpha \in A\}$ is uniformly integrable.* (Dominated $\Rightarrow$ U.I.)

Remk: U.I. $\not\Rightarrow$ Integrable! (Eg. $f(x) = 1$ $\forall x \in \mathbb{R}$, $\{f\}$ is U.I.)

Pf: Note: $\displaystyle\lim_{\lambda\to\infty} \int_{\{F>\lambda\}} F \, d\mu = \lim_{\lambda\to\infty} \int_X 1_{\{F>\lambda\}} F \, d\mu \xrightarrow[\text{(or M.C)}]{\text{D.C.}} 0$

Let $\varepsilon > 0$. Chose $\delta =$ ________

Assume $\mu(E) < \delta$:

$$\int_E |f_n| \, d\mu \le \int_E F = \underbrace{\int_{E\cap\{F>\lambda\}} F} + \int_{E\cap\{F\le\lambda\}} F$$

$$\le \underbrace{\int_{\{F>\lambda\}} F}_{\xrightarrow{\lambda\to\infty} 0} + \lambda\delta$$

Choose λ large s.t. $\int_{\{F>\lambda\}} F < \frac{\varepsilon}{2}$. Choose $\delta \le \frac{\varepsilon}{2\lambda}$ $\Rightarrow$ Done

QED.

Theorem 8.33 (Vitali). *Let $(f_n) \in L^1(X)$. The sequence (f_n) is convergent in L^1 if and only if*

(1) (f_n) converges in measure,
(2) (f_n) is uniformly integrable,
(3) (tightness) $\forall \varepsilon$ *there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\int_{F^c} |f_n| \, d\mu < \varepsilon$ for all n.*

Proof:

(Note: $\mu(X) < \infty \Rightarrow$ Tightness is automatic).

Pf $\Rightarrow$: Assume $(f_n) \longrightarrow f$ in L^1

1. $(f_n) \to f$ in meas (Done $\to$ Chebyshev).

2. $\{f_n\}$ is U. I.

Pf: Pick $\varepsilon > 0$. Want $\delta > 0$ + $\mu(E) < \delta \Rightarrow \forall n \int_E |f_n| < \varepsilon$

$$\int_E |f_n| = \underbrace{\int_E |f_n - f|}_{\leq \int |f_n - f| \\ \leq \varepsilon.} + \underbrace{\int_E |f|}$$

make this $< \varepsilon/2$. $\{f\}$ is U.I. if f is $\mathcal{L}'$

① find n_0 s.t. $\|f_n - f\| < \varepsilon \;\; \forall n > n_0$

② $\{f_1, f_2, \cdots f_{n_0}, f\}$ is U.I. ($\because$ dominated by $\max_{i \leq n_0} |f_i| \vee |f|$)

③ $\Rightarrow \exists \delta_1 > 0$ s.t. if $i \leq n_0$, & $\mu(E) < \delta \Rightarrow \int_E |f_i| < \varepsilon/2$

& $\int_E |f| < \varepsilon/2$.

④ $\Rightarrow \int_E f_n \rightsquigarrow$ ① $n \le n_0 \longrightarrow$ done by $\uparrow$

② $n \ge n_0$: $\int_E |f_n| \le \int_E |f_n - f| + \int_E |f|$

$$\le \underbrace{\int_X |f_n - f|}_{\le \varepsilon/2} + \underbrace{\int_E |f|}_{\le \varepsilon/2}$$

($\because n \ge n_0$)

QED.

③ **Check tightness:** NTS $\forall \varepsilon > 0$, $\exists\, E \ni \mu(E) < \infty$ & $\displaystyle\int_{E^c} |f_n| < \varepsilon \;\; \forall n.$

Scratch: Q1: Show $\forall \varepsilon > 0$, $\exists\, E \ni \mu(E) < \infty$ & $\displaystyle\int_{E^c} |f| < \varepsilon$

$$\left.\begin{aligned} &\mu\{|f| > \delta\} \le \frac{1}{\delta}\|f\|_1 < \infty \quad \forall \delta > 0 \\ &\lim_{\delta \to 0} \int_{|f| \le \delta} |f|\, d\mu \overset{D.C.}{=\!=} 0 \end{aligned}\right\} \Rightarrow \forall \varepsilon > 0,\ \exists\, \delta \ni$$

For $E = \{f > \delta\}$, we have

$\mu(E) < \infty$ & $\displaystyle\int_{E^c} |f|\, d\mu < \varepsilon$

Q^2: If $f_1, f_2 \cdots f_{n_0}$ are finitely many fns,

$$\forall \varepsilon > 0, \ \exists E \ni \mu(E) < \infty \ \& \ \int_{E^c} |f_i| < \varepsilon \quad \forall i \leq n_0$$

Pf of tightness: $\forall \varepsilon > 0, \ \exists n_0 \ni \int_X |f_n - f| < \varepsilon \quad \forall n \geq n_0$.

By above $\exists E \ni \mu(E) < \infty$ & $\forall i \in \{1, \cdots n_0\}$, $\int_{E^c} |f_i| < \varepsilon/2$

& $\int_{E^c} |f| < \varepsilon/2$

$\Rightarrow \forall n \leq n_0 \longrightarrow$ done

$$\forall n \geq n_0, \quad \int_{E^c} |f_n| \leq \underbrace{\int_{E^c} |f_n - f|} + \int_{E^c} |f|$$

$$\leq \underbrace{\int_X |f_n - f|}_{\varepsilon/2} + \varepsilon/2 = \varepsilon$$

QED.

Converse: Say $(f_n) \longrightarrow f$ in meas, $\{f_n\}$ is UI & $\{f_n\}$ is tight. $\Big\}$ NTS $(f_n) \longrightarrow f$ in L^1.

① Assume $f \in L^1$ (You check $\longrightarrow$ not needed).

② Want $\int_X |f_n - f| < \varepsilon$.

$$\int_X |f_n - f| = \underbrace{\int_{\{|f_n - f| \geq \lambda\}}}_{\text{Use U.I. to make this small}} + \underbrace{\int_{\{|f_n - f| < \lambda\}}}$$

$$\int_{\{|f_n-f|<\lambda\}\cap E} |f_n-f| \quad + \quad \int_{\cancel{\{|f_n-f|<\lambda\}}\cap E^c}$$

$$(\text{tightness}) \qquad\qquad\qquad \underbrace{\phantom{\{|f_n-f|<\lambda\}}}_{}$$

$$\underbrace{\phantom{\int_{\{|f_n-f|<\lambda\}\cap E}|f_n-f|}}_{\leq\ \underbrace{\lambda}_{\lambda < \frac{\varepsilon}{3\mu(E)}}\ \mu(E)} \qquad\qquad < \varepsilon \ (\text{by tightness})$$