$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

$$L^p = \{ f \mid \|f\|_p < \infty \}$$

IOU: L^p is a Banach space.

1. $\|f+g\|_p \le \|f\|_p + \|g\|_p$
2. Completeness.

(last time).

Hölder: $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$, $\left| \int_X f g \right| \le \|f\|_p \|g\|_q$.

Lemma 8.23 $\boxed{\text{(Duality)}}$**.** *If* $p \in [1, \infty)$, $1/p + 1/q = 1$, *then* $\|f\|_p = \sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg\, d\mu = \sup_{\|g\|_q = 1} \int_X fg\, d\mu = 1$

Remark 8.24. For $p = \infty$ this is still true if X is σ-finite.

Rmk: If $\sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg\, d\mu < \infty$ then $f \in L^p$ &

$Y \to$ Banach space. $Y^* =$ dual of Y $= \{ y^* : Y \to \mathbb{R} \mid y^*$ is linear & cts$\}$.

Norm on Y^*: define $\|y^*\| = \sup_{\|y\|=1} y^*(y) = \sup_{y \in Y - 0} \frac{y^*(y)}{\|y\|}$

IOU: $(L^p)^* = L^q \quad (1 \le p < \infty)$

Theorem 8.25 (Minkowski's inequality). *If $f, g \in L^p$, then $f+g \in L^p$ and $\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$.*

$(p<\infty)$

Pf: $\|f+g\|_p = \sup_{\|h\|_q=1} \int_X (f+g)\, h \, d\mu \leq \sup_{\|h\|_q=1} \int_X f h \, d\mu + \sup_{\|h\|_q=1} \int_X g h \, d\mu$

$= \|f\|_p + \|g\|_p$ QED.

$p=\infty \to$ easy.

(Note $f, g \in L^p \Rightarrow f+g \in L^p$: Pf $\left(\frac{1}{2}f + \frac{1}{2}g\right)^p \leq \frac{1}{2}|f|^p + \frac{1}{2}|g|^p$

$\Rightarrow |f+g|^p \leq 2^{p-1}\left(|f|^p + |g|^p\right) \leftarrow$ integrable

$\Rightarrow f+g \in L^p$ QED.)

Theorem 8.26 (Jensen's inequality). *If* $\boxed{\mu(X) = 1}$, $\underline{f \in L^1(X)}$, $a < f < b$ *almost everywhere, and* $\varphi : (a, b) \to \mathbb{R}$ *is* $\boxed{\text{convex,}}$ *then*

$$\varphi\Big(\int_X f\,d\mu\Big) \leqslant \int_X \varphi \circ f\,d\mu\,.$$

$\big(a, b \in [-\infty, \infty]\big)$

Bad Proof:

φ convex, $\left.\begin{array}{l} c_i \in [0,1] \text{ \& } \sum_1^N c_i = 1 \\ x_i \in (a,b) \end{array}\right\} \Rightarrow \varphi\left(\sum c_i x_i\right) \leq \sum c_i \varphi(x_i)$

(def of convexity)

$\Rightarrow$ Jensen is true for simple fns!

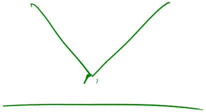
(i.e. $s = \sum a_i \mathbb{1}_{A_i} \Rightarrow \int_X s = \underbrace{\sum a_i \mu(A_i)}_{\text{convex linear comb of } a_i\text{'s.}}$)

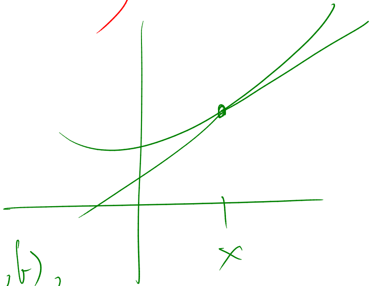
Note $\sum \mu(A_i) = 1$

& use ~~reel~~ ineq with $c_i = \mu(A_i)$. $\Rightarrow$ done

(3) And fns : approximate. (requires some care)

Better Proof: φ convex $\Rightarrow$





$$\varphi(x) + (y - x)\,\varphi'(x) \le \varphi(y) \quad \forall y \in (a,b),\ \forall x \in (a,b)$$

$$\Rightarrow \int_X \left[\varphi(x) + (f(y) - x)\,\varphi'(x)\right] d\mu(y) \le \int_X \varphi(f(y))\, d\mu(y) \quad \forall y \in X$$

$$\Rightarrow \varphi(x) + \varphi'(x) \int_X (f(y) - x)\, d\mu(y) \le \int_X \varphi \circ f(y)\, d\mu(y)$$

Choose $x = \int_X f \, d\mu \Rightarrow \varphi\left(\int_X f \, d\mu\right) + \varphi'(x)\left(0\right) \leq \int_X \varphi \circ f \, d\mu(y)$

QED.

Completeness

Proof of Theorem $\boxed{8.19}$*:* Only remains to show L^p is complete.

Lemma 8.27. *Suppose* $p < \infty$*,* $f_n \in L^p$ *and* $\sum \|f_n\|_p < \infty$*. Let* $f = \sum f_n$*. Then* $f \in L^p$*, and* $\sum f_n \to f$ *in* L^p *and* $\sum f_n \to f$ *almost everywhere.*

Pf: ① Let $F = \sum_1^\infty |f_n|$.

Let $S_N = \sum_1^N |f_n| \in L^p$. Know $\|S_N\|_p \leq \sum_1^N \|f_n\|_p \to \sum_1^\infty \|f_n\|_p < \infty$

$\Rightarrow \|S_N\|_p^p \to \left(\sum_1^\infty \|f_n\|_p\right)^p < \infty.$

$\int_X |S_N|^p \, d\mu$

$\Rightarrow F \in L^p \quad \left(\because \int_X F^p \, d\mu \overset{\text{M.C.}}{=} \lim_{N\to\infty} \int_X S_N^p \, d\mu < \infty\right)$

② $\Rightarrow f \in L^p$, $\Rightarrow \sum_1^\infty |f_n| < \infty$ a.e.

$\Rightarrow \sum_1^\infty f_n$ is cgt a.e. $\quad$ ($\Rightarrow$ ~~claim~~ 2nd assertion).

③ Let $f = \sum_1^\infty f_n$. NTS $\left(\sum_1^N f_n\right) \longrightarrow f$ in L^p.

Note $f - \sum_1^N f_n = \sum_{N+1}^\infty f_n$

$$\Rightarrow \left\| f - \sum_1^N f_n \right\|_p \leq \sum_{N+1}^\infty \|f_n\|_p \xrightarrow{N \to \infty} 0$$

QED.

Proof of Theorem 8.19: (L^p is complete).

Pf: Suppose (f_n) is a Cauchy seq in L^p

$\exists\, n_k \ni \|f_{n_{k+1}} - f_{n_k}\| \leq 2^{-k}$

By lemma, $\sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$ is cgt in L^p

$\Leftrightarrow (f_{n_k})$ is cgt in L^p.

Since (f_n) is Cauchy & (f_{n_k}) is cgt $\Rightarrow (f_n)$ is cgt QED.

HW 5: Q 6

① $U = \bigcup_{1}^{\infty} (a_k, b_k)$

② $\lim_{n \to \infty} (a_k, b_k) \cap A_n = \left.\dfrac{b_k - a_k}{2}.\right\}$?

$\Rightarrow \lim_{n \to \infty} \sum_{1}^{\infty} (a_k, b_k) \cap A_n$

$\|$

$\underline{\left(\sum_{k=1}^{\infty} \dfrac{b_k - a_k}{2} \right.}$