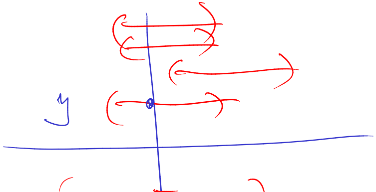
Example of a regular meas in a non σ-finite space

$X = \mathbb{R}$ (λ meas)

$Y = \mathbb{R}$ (counting meas, discrete top)

$X \times Y \longrightarrow$ open sets



If $\lambda(U_y) > 0$ for uncountably many y, then $\Sigma(\ ) = \infty$

μ: $U \subseteq X \times Y$ open

$U = \bigcup_{y \in \mathbb{R}} U_y \times \{y\}$ ($U_y \subseteq X$ open) define $\mu(U) = \sum_{y \in \mathbb{R}} \lambda(U_y)$

$A \subseteq X \times Y$ Borel, $\mu(A) = \inf\{\mu(U) \mid U \supseteq A \text{ open}\}$.

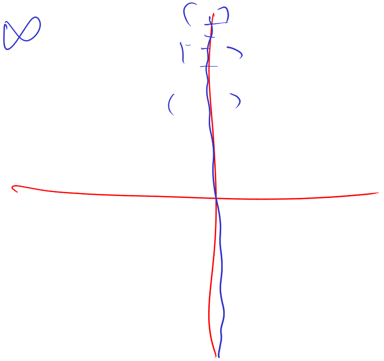
You check μ gives a regular measure on $X \times Y$.

Q: What is the measure of $\{0\} \times Y = \infty$

Q: $K \subseteq \{0\} \times Y$ is cpt.

What is $\mu(K) = 0$

(Gives necessary C-eg on HW 5/6).



Definition 8.16. Define an equivalence relation on $\mathcal{L}^p$ by $\boxed{f \sim g}$ if $f = g$ almost everywhere.

Definition 8.17. Define $\mathcal{L}^p(X) = \mathcal{L}^p(X)/\sim$.

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

Theorem 8.19. *For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.*

$$\|f\|_p = \left(\int_X |f|^p \right)^{1/p}.$$

Theorem 8.20 (Hölder's inequality)**.** *Say $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $|\int_X fg\, d\mu| \leqslant \|f\|_p \|g\|_q$.*

Remark 8.21. The relation between p and q can be $\boxed{\text{motivated by dimension counting}}$, or scaling.

Let $\varepsilon > 0$. $X = \mathbb{R}^d$. $f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$, $g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$

① $\int_{\mathbb{R}^d} f_\varepsilon g_\varepsilon = \varepsilon^d \int_{\mathbb{R}^d} f\left(\frac{x}{\varepsilon}\right) g\left(\frac{x}{\varepsilon}\right) \frac{dx}{\varepsilon^d} = \varepsilon^d \int_{\mathbb{R}^d} f g \, d\lambda.$

② $\|f_\varepsilon\|_p = \left(\varepsilon^d \int_{\mathbb{R}^d} \left|f\left(\frac{x}{\varepsilon}\right)\right|^p \frac{dx}{\varepsilon^d}\right)^{1/p} = \varepsilon^{d/p} \|f\|_p$

③ $\|g\|_q = \ldots = \varepsilon^{d/q} \|g\|_q$

If Hölder is true $\Rightarrow$ $\left| \int_{\mathbb{R}^d} f_\varepsilon g_\varepsilon \, dx \right| \leq \|f_\varepsilon\|_p \, \|g_\varepsilon\|_q$

$$\varepsilon^d \left| \int_{\mathbb{R}^d} f g \right| \qquad\qquad \varepsilon^{\frac{d}{p} + d/q} \|f\|_p \, \|g\|_q$$

Since this is true $\forall \varepsilon$, must have $d = \frac{d}{p} + \frac{d}{q} \Leftrightarrow \boxed{\frac{1}{p} + \frac{1}{q} = 1}$

Brute force proof of Theorem 8.20 :

Stupid.

$\Rightarrow$ ① Induction $\sum_1^N x_i y_i \le \left(\sum_1^N x_i^p\right)^{1/p} \left(\sum_1^N y_i^q\right)^{1/q}$

$(x_i, y_i \ge 0)$.

② Say $c_i \ge 0$, $\sum x_i y_i c_i = \sum \underbrace{x_i c_i^{1/p}}\, y_i c_i^{1/q}$

$\le \left(\sum x_i^p c_i\right)^{1/p} \left(\sum y_i^q c_i\right)^{1/q}$

③ $\Rightarrow$ Holder is true for simple fns.

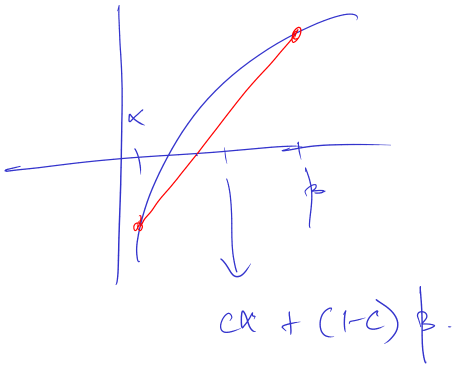
④ Approximate $\Rightarrow$ QED.

Proof of Theorem 8.20 using Young's inequality.

Theorem 8.22 (Young's inequality). *If $x, y \geqslant 0$, $\boxed{1/p + 1/q = 1}$ then $xy \leqslant x^p/p + y^q/q$.*

Pf 1: Calculus min ( )

Pf 2: $\ln x$ is concave & increasing



$\Rightarrow c \in (0,1), \quad \alpha, \beta > 0$

$$c \ln \alpha + (1-c) \ln \beta \leq \ln(c\alpha + (1-c)\beta)$$

$$\ln(\alpha^c \beta^{1-c})$$

$$\Rightarrow \quad \alpha^c \beta^{1-c} \leq c\alpha + (1-c)\beta.$$

$c = \frac{1}{p}. \quad 1-c = \frac{1}{q}.$

$\alpha = x^p \quad \beta = y^q$

QED.

Pf of Holder: NTS $\left|\int_X fg\right| \le \|f\|_p \|g\|_q$.

Let $\tilde{f} = \dfrac{f}{\|f\|_p}$ & $\tilde{g} = \dfrac{g}{\|g\|_q}$.

$(\|\tilde{f}\|_p = 1 \qquad \|\tilde{g}\|_q = 1)$

Holder $\Leftrightarrow$ showing $\left|\int_X \tilde{f}\tilde{g}\right| \le 1$

Pf of :

$$\left|\int_X \tilde{f}\,\tilde{g}\,d\mu\right| \le \int_X |\tilde{f}||\tilde{g}|\,d\mu \overset{\text{Young}}{\le} \int_X \left(\frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q}\right) d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

QED.

(You check $p = 1$, $q = \infty$)

Lemma 8.23 (Duality). *If* $p \in [1,\infty)$, $1/p + 1/q = 1$, *then* $\|f\|_p = \sup_{g\in L^q - 0} \frac{1}{\|g\|_q}\int_X fg\,d\mu = \sup_{\|g\|_q = 1}\int_X fg\,d\mu$

Remark 8.24. For $p=\infty$ this is still true if X is σ-finite.

Pf of Duality: 1 $\frac{1}{\|g\|_q}\int_X fg\,d\mu \le \|f\|_p$ (Holder).

$\Rightarrow \sup_{g\in L^q - \{0\}} \frac{1}{\|g\|_q}\int_X fg\,d\mu \le \|f\|_p$.

2 NTS equality. Chose $g = |f|^{p-1}\operatorname{sign}(f) \Rightarrow fg = |f|^p$.

$\|g\|_q^q = \int_X |g|^q\,d\mu = \int_X |f|^{pq-q} = \int_X |f|^p$

$$\boxed{\Leftrightarrow \|g\|_q^q = \|f\|_p^p}$$

$$\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow pq = p+q$$

$$\Rightarrow \frac{1}{\|g\|_q}\int_X fg \, d\mu = \frac{1}{\|g\|_q}\int_X |f|^p \, d\mu = \frac{\|f\|^p}{\|g\|_q} = \|f\|_p^{p\left(1-\frac{1}{q}\right)} = \|f\|_p$$

QED.

Theorem 8.25 (Minkowski's inequality). *If* $f, g \in L^p$, *then* $f + g \in L^p$ *and* $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.