Theorem 8.5. *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty,}$ then $(f_n) \to f$ in measure.*

Lemma 8.6 (Egorov). *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty,}$ for every $\varepsilon > 0$ there exists A_ε such that $\mu(A_\varepsilon^c) < \varepsilon$ and $(f_n) \to f$ uniformly on A_ε.*

Question 8.7. *Does this imply $(f_n) \to f$ uniformly almost everywhere?*

$(f_n) \xrightarrow{\mu} f$

if $\forall \varepsilon > 0$

$\mu(|f_n - f| > \varepsilon) \xrightarrow{n\to\infty} 0$

$\to$ Egorov $\to$ last time.

Thm 8.5: Pick $\varepsilon > 0$. NTS $\mu(|f_n - f| > \varepsilon) \longrightarrow 0$

Egorov $\Rightarrow$ $\forall \delta > 0$, $\exists A_\delta$ s.t. $\mu(A_\delta^c) < \delta$

& $f_n \to f$ unif on A_δ.

$\Rightarrow \{|f_n - f| > \varepsilon\} \subseteq A_\delta^c$ $\forall$ large n.

Proof of Theorem 8.5

$$\Rightarrow \mu(\{|f_n - f| > \varepsilon\}) \le \mu(A_\delta^c) = \delta$$

QED.

$$\begin{tikzcd}
 & f_n \to f \ \text{a.e.} \arrow[dl, Rightarrow, "(\mu(X)<\infty)"'] \arrow[dr, Rightarrow, "/" marking] & \\
f_n \to f \ \text{meas} \arrow[ur, Rightarrow, bend left, "\text{along s.s.}"] \arrow[rr, Rightarrow, bend right, "/" marking] & & f_n \to f \ L^p \arrow[ll, Rightarrow, "(p<\infty)"] \arrow[ul, Rightarrow, bend right, "\text{alg s.s.}"']
\end{tikzcd}$$

Proposition 8.8. *If $(f_n) \to f$ in measure then (f_n) need not converge to f almost everywhere.*

Eg: Pf by Picture $f_1 = \mathbb{1}_{[0, 1/2]}$ $f_2 = \mathbb{1}_{[1/2, 1]}$

$f_3 = \mathbb{1}_{[0, 1/4]}$, $f_4 = \mathbb{1}_{[1/4, 1/2]}$, $f_5 = \mathbb{1}_{[1/2, 3/4]}$ $f_6 = \mathbb{1}_{[3/4, 1]}$

$f_7 = \mathbb{1}_{[0, 1/8]}$, ... $f_8 = \mathbb{1}_{[1/8, \frac{2}{8}]}$ etc.

Q: $(f_n) \longrightarrow 0$ in measure $\left(\mu(|f_n - 0| > \varepsilon) \longrightarrow 0 \ \forall \varepsilon\right)$

Q: $(f_n) \longrightarrow 0$ ptwise? NO. $\forall x$ $f_n(x) = 0$ i.o. & $f_n(x) = 1$ i.o.

Proposition 8.9. *If $(f_n) \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere.*

Pf: $\forall k \in \mathbb{N}$, $\mu(|f_n - f| > \frac{1}{k}) \xrightarrow{n\to\infty} 0$

$\forall k$, $\exists n_k \ni n_k > n_{k-1}$ & $\mu(|f_{n_k} - f| > \frac{1}{k}) \leq \frac{1}{2^k}$

Let $A_k = \{|f_{n_k} - f| > \frac{1}{k}\}$.

Let $B = \{x \mid x$ only $\in$ finitely many $A_k\}$

① $\forall x \in B$, $|f_{n_k}(x) - f(x)| \leq \frac{1}{k}$ $\forall$ large k $\Rightarrow (f_{n_k}(x)) \longrightarrow f(x)$

② NTS $\mu(B^c) = 0$

$$B^c = \{x \mid x \in \infty^{ly} \text{ many } A_k\}$$

$$= \{x \mid \forall m\ \exists n \geq m \ \ni \ x \in A_n\}$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n$$

$$\Rightarrow \forall m,\ \mu(B^c) \leq \mu\left(\bigcup_{n \geq m} A_n\right) \leq \sum_{n \geq m} \mu(A_n) = \sum_{n \geq m} \frac{1}{2^n} = \frac{1}{2^m}$$

$$\mu(B^c) \leq \frac{1}{2^m} \quad \forall m \quad \Rightarrow \mu(B^c) = 0$$

QED.

8.2. L^p spaces.

Definition 8.10. A $\boxed{\textit{Banach space}}$ is a normed vector space that is complete under the metric induced by the norm.

Example 8.11. $\mathbb{C}$, $\mathbb{R}^d$, $C(X)$, etc. are all Banach spaces.

$C(X) = \{f : X \to \mathbb{R} \text{ cts}\}$, X cpt.

Definition 8.12. For $p \in (0, \infty)$, define $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$.

$\|f\| = \sup_{x \in X} |f(x)|$

Definition 8.13. For $p = \infty$, define $\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\{C \geqslant 0 \mid |f| \leqslant C$ almost ~~surely~~ everywhere$\}$

Definition 8.14. Let (X, Σ, μ) be a measure space, and assume Σ is μ-complete. Define $\mathcal{L}^p(X) = \{f : X \to \mathbb{R} \mid \|f\|_p < \infty\}$.

Question 8.15. *Is $\mathcal{L}^p(X)$ a Banach space?*

$X \to$ normed V.S.

$\|\cdot\| : X \to [0, \infty)$ s.t.

(0) $\|x\| = 0 \iff x = 0$

(1) $\|\lambda x\| = |\lambda| \, \|x\|$

& (2) $\|x + y\| \leq \|x\| + \|y\|$

dist: $d(x, y) = \|x - y\|$

(X, d) metric

$p=\infty$: $\|f\|_\infty =$ " $\sup_{x\in X} |f(x)|$ " $= \sup\{a \mid \mu\{|f|\geq a\} > 0\}$.

ess sup

$\longrightarrow = \inf\{c \mid |f|\leq c \text{ a.e.}\}$

$\left(-\|f\|_\infty \leq f \leq \|f\|_\infty \text{ a.e.}\right)$.

$\mathcal{L}^p(X) = \{f \mid \|f\|_p < \infty\}$. $\leftarrow$ Not a Banach space.

$\underline{f=0 \text{ a.e.}}$, but $f\neq 0$, we still have $\|f\|_p = 0$

Definition 8.16. Define an equivalence relation on $\mathcal{L}^p$ by $f \sim g$ if $f = g$ almost everywhere.

Definition 8.17. Define $L^p(X) = \mathcal{L}^p(X)/\sim$.

$$L^p(X) = \mathcal{L}^p(X)/\sim$$

Remark 8.18. We will always treat elements of $L^p(X)$ as functions, implicitly $\boxed{\text{identifying a function with its equivalence class}}$ under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

$$\boxed{\textbf{Theorem 8.19. } \textit{For } p \in [1, \infty],\ L^p(X) \textit{ is a Banach space.}}$$

IOU Proof.
Hard details to check
1. $\triangle$ ineq
2. Completeness

$$L^p(X) = \{ f \mid \|f\|_p < \infty \}$$

$f \in L^p$. implicitly mean class of all g ∋ $g = f$ a.e.

$\hookrightarrow f(x_0)$, $x_0 \in X$. $f(x_0) \longrightarrow$ not defined on L^p.

$f \in L^p \longrightarrow \int_A f \, d\mu \leftarrow$ OK, $\mu\{|f| > \lambda\} \leftarrow$ OK

Theorem 8.20 (Hölder's inequality)**.** *Say $p, q \in [1, \infty]$ with $\boxed{1/p + 1/q = 1}$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and* $|\int_X fg \, d\mu| \leqslant \|f\|_p \|g\|_q$.

Hölder conjugates.

Remark 8.21. The relation between p and q can be motivated by dimension counting, or scaling.

Motivation $\longrightarrow$ "Dimension counting".

$f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ (dimensionless)

$L \to$ length (dimension).

Q: dim of $\int_{\mathbb{R}^d} f g \, d\lambda = L^d$

$\dim \|f\|_p = \dim \left(\int_{\mathbb{R}^d} |f|^p dx \right)^{1/p}$

$= L^{d/p}$

$\dim \|f\|_q = L^{d/q}$

$\dim (\|f\|_p \, \|g\|_q) = L^{\frac{d}{p} + \frac{d}{q}}$

$= d$