Last time: $\underbrace{\overset{MC}{DCT}}$

$(f_n) \to f$ a.e.

$|f_n| \le F$ a.e. $\forall n$ (F ind of n)

& $\int_X F \, d\mu < \infty$

Then $\lim \int_X f_n \, d\mu = \int_X f \, d\mu$.

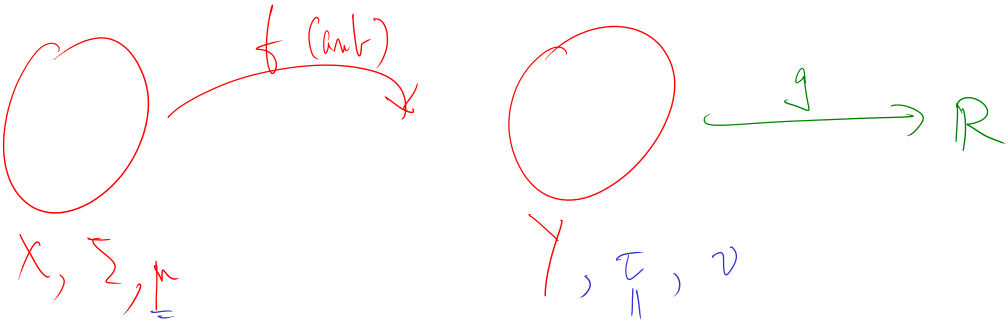
7.3. Push forward measures.

Definition 7.21. Say $f\colon X \to \mathbb{R}^d$ is integrable, then define $\int_X f\,d\mu = (\int_X f_1\,d\mu, \ldots, \int_X f_d\,d\mu$, where $f = (f_1, \ldots, f_d)$.

Theorem 7.22. *Let (X, Σ, μ) be a measure space, $f\colon X \to Y$ be arbitrary. Define $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$, and define $\nu(A) = \mu(f^{-1}(A))$. Then ν is a measure on (Y, τ) and $\int_Y g\,d\nu = \int_X g \circ f\,d\mu$.*

Remark 7.23. The measure ν is called the *push forward* of μ and denoted by $f^*(\mu)$, or $\mu_{f^{-1}}$. This is used often to define Laws of random variables. (We will use it to prove the change of variable formula.)

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right), \quad A_i \text{ disj}$$
$$= \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right)$$
$$= \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(A_i)\right)$$
$$= \sum_{i=1}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=1}^{\infty} \nu(A_i)$$



$$\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$$

Prop: Given $g : Y \to \mathbb{R}$. $\int_Y g \, d\nu = \int_X g \circ f \, d\mu$

Pf: Suppose $s : Y \to \mathbb{R}$ is simple.

$$s = \sum a_i \mathbb{1}_{A_i} \Rightarrow \int_Y s \, d\nu = \sum a_i \nu(A_i) = \sum a_i \mu(f^{-1}(A))$$

Also, $\int_X (s \circ f) \, d\mu = \int_X \sum a_i \mathbb{1}_{f^{-1}(A_i)} \, d\mu =$

$\Rightarrow \forall s$ simple, $\int_Y s \, d\nu = \int_X (s \circ f) \, d\mu$.

If $g : Y \to \mathbb{R}$ is ≥ 0

then find s_n simple $\ni (s_n) \to g$, $0 \leq s_n \leq s_{n+1}$

$$\Rightarrow \int_Y g\,d\nu \overset{MC}{=} \lim \int_Y s_n\,d\nu = \lim \int_X (s_n \circ f)\,d\nu \overset{M.C.}{=} \int_X (g \circ f)\,d\mu.$$

QED.

Corollary 7.24. *If* $\alpha \in \mathbb{R}^d$, *then* $\int_{\mathbb{R}^d} f(x+\alpha)\,d\lambda(x) = \int_{\mathbb{R}^d} f(x)\,d\lambda(x)$.

$\hookrightarrow g : \mathbb{R}^d \to \mathbb{R}^d \qquad g(x) = x + \alpha$

Then $g^*(\lambda) = \lambda.$

By thm
$$\int_{\mathbb{R}^d} f \circ g \, d\lambda = \int_{\mathbb{R}} f \, d(\underbrace{g^* \lambda}_{=\lambda}) = \int_{\mathbb{R}^d} f(x)\, d\lambda(x)$$

$$\| $$

$$\int_{\mathbb{R}^d} f(x+\alpha)\, d\lambda(x)$$

QED.

8. Convergence

8.1. Modes of convergence.

Definition 8.1. We say $(f_n) \to f$ almost everywhere if for almost every $x \in X$, we have $(f_n(x)) \to f(x)$.

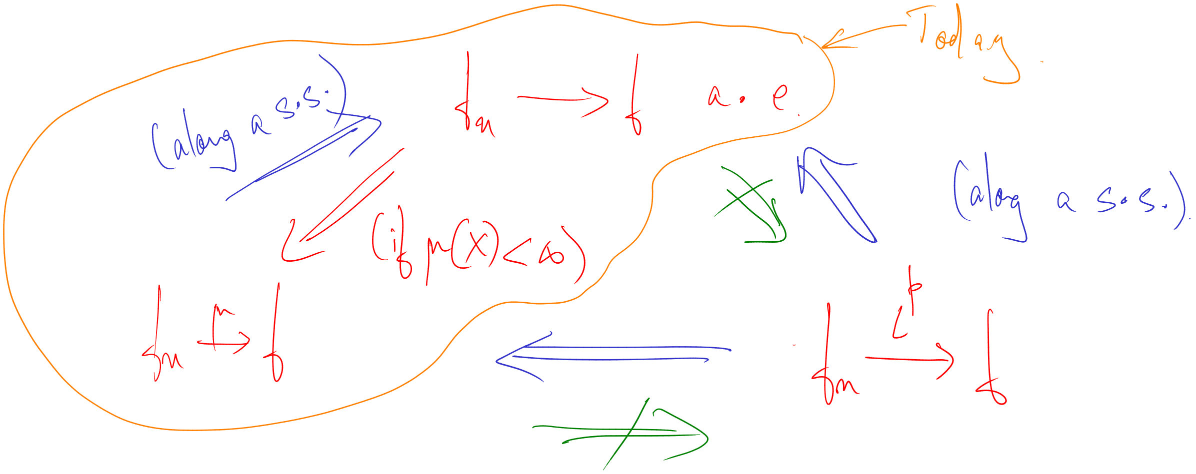
Definition 8.2. We say $(f_n) \to f$ in measure (notation $(f_n) \xrightarrow{\mu} f$) if for all $\varepsilon > 0$, we have $(\mu\{|f_n - f| > \varepsilon\}) \to 0$.

Definition 8.3. Let $p \in [1, \infty)$. We say $(f_n) \to f$ in L^p if $(\int_X |f_n - f|^p \, d\mu) \to 0$.

Question 8.4. *Why $p > 1$? How about $p = \infty$?*

$$\lim_{n\to\infty} \Big(\underbrace{\int |f_n - f|^p \, d\mu}_{\text{"dist between } f_n \text{ \& } f\text{"}} \Big)^{1/p} = 0$$

(1) $(f_n) \to f$ almost everywhere implies $(f_n) \to f$ in measure if $\mu(X) < \infty$.
(2) $(f_n) \to f$ in measure implies $(f_n) \to f$ almost everywhere along a subsequence.
(3) $(f_n) \to f$ in L^p implies $(f_n) \to f$ in measure (for $p < \infty$), and hence $(f_n) \to f$ along a subsequence.
(4) Convergence almost everywhere or in measure don't imply convergence in L^p.



Eg: $(f_n) \longrightarrow f$ a.e. but $(f_n) \not\longrightarrow f$ in meas

Chose $f_n = \mathbb{1}_{[n,\infty]}$, $f = 0$

$$\left\{ \begin{array}{l} (f_n) \to 0 \text{ a.e.} \\ (f_n) \not\to 0 \text{ in meas} \end{array} \right.$$

$\left(\because \ \{ |f_n - f| > \frac{1}{2} \} = \infty \ \forall n . \right)$

Theorem 8.5. *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty}$, then $(f_n) \to f$ in measure.*

Lemma 8.6 (Egorov). *If $(f_n) \to f$ almost everywhere and $\boxed{\mu(X) < \infty}$, for every $\varepsilon > 0$ there exists A_ε such that $(f_n) \to f$ uniformly on* ~~A_ε^c~~. A_ε & $\mu(A_\varepsilon^c) < \varepsilon$

Question 8.7. *Does this imply $(f_n) \to f$ uniformly almost everywhere?* (No: $f_n(x) = x^n$ $[0,1]$.)

Pf of Egorov: $\forall k \in \mathbb{N}$, $\bigcup_{m=1}^{\infty} \bigcap_{n \geq m} \{ |f_n - f| \leq \frac{1}{k} \} = X -$ null set

& this is an inc union. $\Rightarrow \lim_{m \to \infty} \mu\left(\bigcap_{n \geq m} \{ |f_n - f| < \frac{1}{k} \} \right) = \mu(X)$

$\Rightarrow \forall k, \exists n_k \ni \mu\Big(\underbrace{\bigcap_{n \geq n_k} \{ |f_n - f| < \frac{1}{k} \}}_{A_k} \Big) \geq \mu(X) - \frac{\varepsilon}{2^k}$

Let $A = \bigcap_{k=1}^{\infty} A_k$. ① Note $\mu(A^c) \leq \sum_1^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$

② Note : $f_n \longrightarrow f$ unif on A.

$\left(\because \forall n \geq n_k, \ |f_n - f| \leq \frac{1}{k} \ \forall x \in A_k \supseteq A \right)$

QED.