Last time: $\int_X f\,d\mu = \underbrace{\int_X f^+\,d\mu} - \int_X f^-\,d\mu.$

$f \geq 0$: $\int_X f\,d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_X s\,d\mu$ $\qquad s = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$ & define $\int_X s\,d\mu = \sum a_k \mu(A_k)$

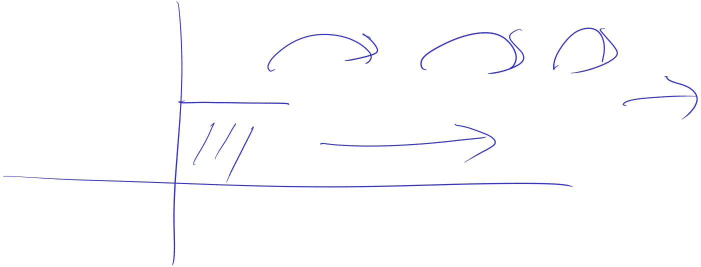
M.C.: $0 \leq f_n \leq f_{n+1}$, $\underbrace{(f_n)} \to \underline{f} \Rightarrow \int_X f_n\,d\mu \to \int_X f\,d\mu.$

7.2. **Dominated convergence.** When does $\lim \int_X f_n \, d\mu \neq \int_X f \, d\mu$? Two typical situations where it fails:

(1) Mass escapes to infinity

(2) Mass clusters at a point

$$f_n(x) = \mathbb{1}_{[n,\, n+1]}$$



$(f_n) \longrightarrow 0$ But $\int_{\mathbb{R}} f_n \, dx = 1$

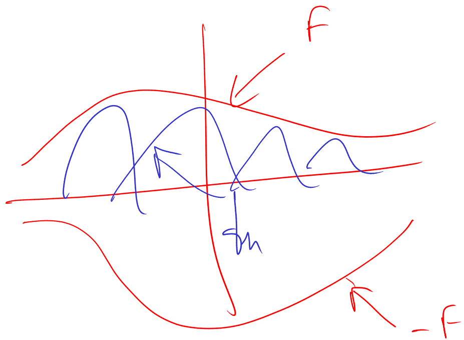
$$f_n(x) = n \, \mathbb{1}_{[0, \frac{1}{n}]}$$

① $(f_n) \longrightarrow 0$ (a.e.)

② $\int_{\mathbb{R}} f_n \, dx = 1 \quad \forall n$

$\Rightarrow \lim \int_{\mathbb{R}} f_n \, dx \neq \int_{\mathbb{R}} f \, dx = 0$

Theorem 7.15 (Dominated convergence)**.** *Say (f_n) is a sequence of measurable functions, such that $(f_n) \to f$ almost everywhere. Moreover, there exists $F \in L^1(X)$ such that $|f_n| \leqslant F$ almost everywhere. Then* $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.



Lemma 7.16 (Fatou). *Suppose* $f_n \geqslant 0$, *and* $(f_n) \xrightarrow{(a.e.)} f$. *Then* $\liminf \int_X f_n \, d\mu \geqslant \int_X f \, d\mu$.

(+ve fns → mass can escape, but not be created)

Pf: Let $g_n = \inf_{k \geq n} f_k$, Note $0 \leq g_n \leq g_{n+1}$

$\Rightarrow$ By M.C. $\lim_{n \to \infty} \int_X g_n \, d\mu = \int_X \left(\lim_{n \to \infty} g_n\right) d\mu = \int_X f \, d\mu.$

But $g_n \leq f_n \Rightarrow \underbrace{\int_X g_n \, d\mu}_{\downarrow\, n \to \infty} \leq \int_X f_n \, d\mu$

$\int_X f \, d\mu \longrightarrow$ QED.

Proof of Theorem $\boxed{7.15}$ D.C. $(f_n) \to f$, $|f_n| \le F \in L^1(X)$

NTS $\lim \int_X f_n \, d\mu = \int_X f \, d\mu$. $\int_X F \, d\mu < \infty$.

Pf: ① Let $g_n = F + f_n \ge 0$

By Fatou: $\underline{\lim} \int_X g_n \, d\mu \ge \int_X (F+f) \, d\mu$

$\underline{\lim} \int_X (F + f_n) \, d\mu$

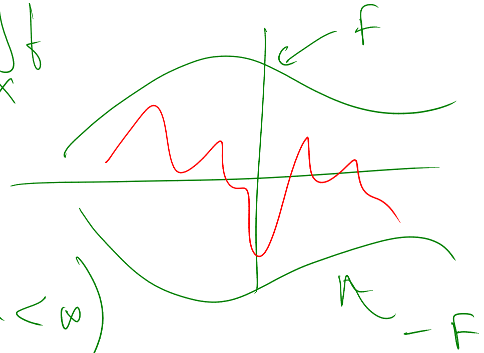
$\Rightarrow \underline{\lim} \int_X f_n \, d\mu \ge \int_X f \, d\mu$

$\left(\because \int_X F \, d\mu < \infty\right)$.

② $h_n = F - f_n \ge 0$

Fatou $\Rightarrow$ $\displaystyle \underline{\lim} \int_X (F - f_n) \geq \int_X F - \int_X f$

$\displaystyle \Rightarrow \overline{\lim} \int_X f_n \, d\mu \leq \int_X f \, d\mu$



$\left(\because \int_X F \, d\mu < \infty \right)$

QED.

Theorem 7.17 (Beppo-Levi). *If* $f_n \geqslant 0$, *then* $\sum_1^\infty \int_X f_n \, d\mu = \int_X (\sum_1^\infty f_n) \, d\mu$.

Pf: $s_n = \sum_1^n f_k$ & use M. C.

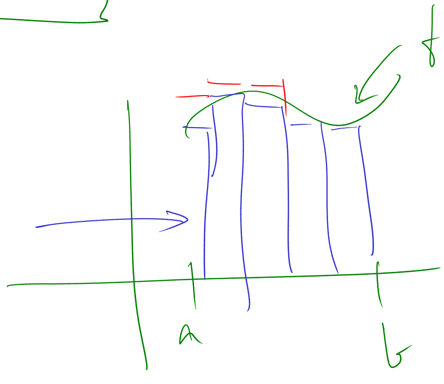
Theorem 7.18. *If $f : \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, then the Riemann integral of f is the same as the Lebesgue integral.*

Proof. IOU $\square$

HW.

ε wiggling

lower sum
$=$
$\int$ simple fn.



Question 7.19. *Let $f : [0,\infty) \to \mathbb{R}$ be measurable, and define the* Laplace transform *of f by $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Is F continuous? Is F differentiable?*

$f(t)$ $\qquad F(s)$ $\qquad \int_0^\infty e^{-st} f(t)\, d\lambda(t)$

Lebesgue.

Q1: Is F cts?

$s_n \to s$ (some seq). Want $F(s_n) \to F(s)$

Want $\int_0^\infty e^{-s_n t} f(t)\,dt \longrightarrow \int_0^\infty e^{-st} f(t)\,dt$

Clearly $\left(e^{-s_n t} f(t)\right) \longrightarrow \left(e^{-st} f(t)\right) \quad \forall t$

$\boxed{\text{Say } \int_0^\infty |f|\,dx < \infty.}$ $\qquad$ Note: $\left|e^{-s_n t} f(t)\right| \le |f(t)| \quad \forall n$

D.C. $\Rightarrow$ $$\lim \int_0^\infty e^{-s_n t} f(t)\, dt = \int_0^\infty e^{-st} f(t)\, dt.$$

$$\boxed{\therefore \text{ 1 If } f \in L^1 \Rightarrow F \text{ is cts!}}$$

Q2: Is F diff? s

Pick $s > 0$, $(s_n) \to s$.

$$\frac{F(s_n) - F(s)}{s_n - s} = \int_0^\infty \underbrace{\left(\frac{e^{-s_n t} - e^{-st}}{s_n - s} \right) f(t)}_{g_n} dt$$

Let $g_n(t) = \dfrac{e^{-s_n t} - e^{-st}}{s_n - s} f(t)$

Note $\left(g_n(t) \right) \longrightarrow -t e^{-st} f(t)$

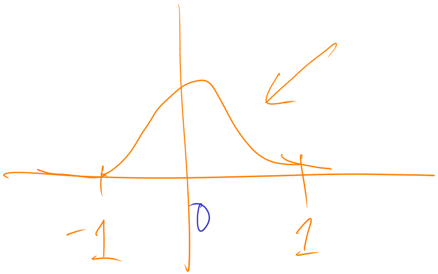
Q: Does $\int_0^\infty g_n(t)\,dt \longrightarrow -\int_0^\infty t e^{-st} f(t)\,dt$ $\left(\Rightarrow \underline{F} \text{ is diff at } s \right)$

Note $|g_n(t)| \leq \underbrace{\left| \dfrac{e^{-s_n t} - e^{-st}}{s_n - s} \right|}$ $|f(t)|$

$\leq t\,|f(t)|$ (M.V.T).

$\boxed{\begin{array}{l} \text{If } t f(t) \in L^1 \\ \Rightarrow \text{D.C. } F \text{ is diff} \end{array}}$

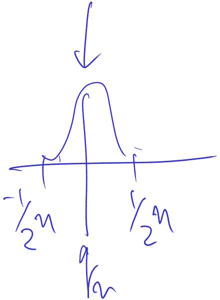
Question 7.20. *Let φ be a bump function, and (q_n) be an enumeration of the rationals. Define $f(x) = \sum_{n=1}^{\infty} \varphi(2^n(x - q_n))$. Is f finite almost everywhere?*



$\varphi \geq 0$

φ : cptly supp

$\int_{-\infty}^{\infty} \varphi = 1$.



$\int \varphi$

$$\int f\,dx \overset{BL}{=} \sum_{1}^{\infty} \underbrace{\int \varphi(2^n(x-q_n))}$$

$$= \sum_{1}^{\infty} \frac{1}{2^n} < \infty$$

$\Rightarrow f < \infty$ a.e.!!