$\left(s = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}\right)$

7. Integration

7.1. Construction of the Lebesgue integral. Recall, $s\colon X \to \mathbb{R}$ is *simple* if s is measurable and has finite range.

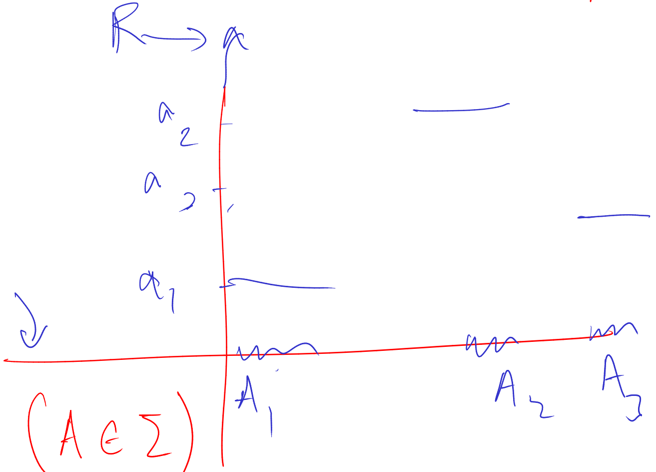
Definition 7.1. Let $s \geqslant 0$ be a simple function. Let $\{a_1, \dots, a_n\} = s(X)$, and set $A_i = s^{-1}(a_i)$. Define $\int_X s\,d\mu = \sum_{i=1}^n a_i \cancel{A_i}\,\mu(A_i)$.

Remark 7.2. Always use the convention $0 \cdot \infty = 0$.

Remark 7.3. Other notation: $\int_X s\,d\mu = \int_X s(x)\,d\mu(x)$.

$$\int_X s\,d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

could be ∞



$$\int s = \int_X s\,d\mu$$

For MT only

Notation: $A \subseteq X$ $(A \in \Sigma)$

$$\int_A s\,d\mu = \int_X \mathbb{1}_A s\,d\mu = \sum_{i=1}^{n} a_i \mu(A \cap A_i)$$

Proposition 7.4. *If* $0 \leqslant s \leqslant t$ *are simple, then* $\int_X s\,d\mu \leqslant \int_X t\,d\mu$. (monotonicity of μ)

Proposition 7.5. *If* $s, t \geqslant 0$ *are simple, then* $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$.

(Same checking involved, straightforward)

Definition 7.6. Let $f\colon X \to [0,\infty]$ be measurable. Define $\int_X f\,d\mu = \sup\{\int_X s\,d\mu \mid 0 \leqslant s \leqslant f,\ s \text{ simple.}\}$. $(f \geq 0)$

Definition 7.7. Let $f\colon X \to [-\infty,\infty]$ be measurable. We say f is *integrable* if $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$. In this case we define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$. (lose cancellation!)

Definition 7.8. We let $L^1(X) = L^1(X,\Sigma,\mu)$ be the set of all integrable functions on X. (Note $f \in L^1 \iff |f| \in L^1$.)

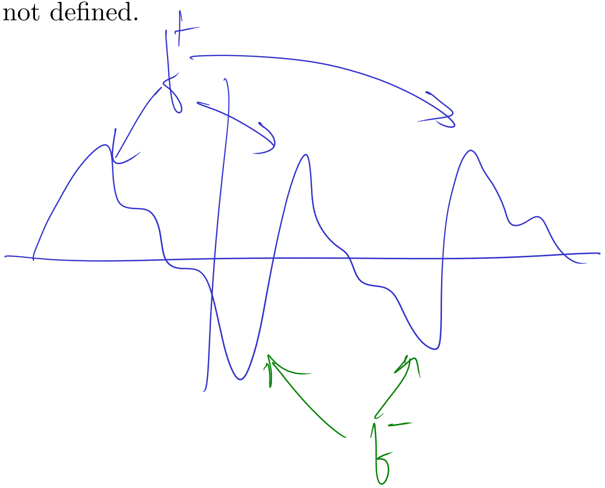
Definition 7.9. We say f is *integrable in the extended sense* if *either* $\int_X f^+\,d\mu < \infty$ or $\int_X f^-\,d\mu < \infty$. In this case we still define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Remark 7.10. If both $\int_X f^+\,d\mu = \infty$ and $\int_X f^-\,d\mu = \infty$, then $\int_X f\,d\mu$ is not defined.

$\boxed{\textbf{Question 7.11. } \textit{Do we have linearity?}}$

$$f^+ = f \vee 0 = \max\{f, 0\}$$

$$f^- = -f \wedge 0 = -\min\{f, 0\}$$



Check linearity: $\quad f, g \geq 0$.

$0 \leq s \leq f$, $0 \leq t \leq g$, $\quad s, t$ simple,

$\Rightarrow 0 \leq s + t \leq f + g$

$\Rightarrow \int_X (f+g)\,d\mu \geq \sup\left\{ \int_X (s+t)\,d\mu \;\middle|\; \begin{array}{l} 0 \leq s \leq f \\ 0 \leq t \leq g \end{array}, \; s, t \text{ simple} \right\}$

$$\| $$

$$\int_X f\,d\mu + \int_X g\,d\mu.$$

Rmk. If $\mu(X) < \infty$ & f, g bdd can show $\int_X (f+g)\,d\mu \leq \int_X f\,d\mu + \int_X g\,d\mu$

Proposition 7.12 (Consistency). *If* $s = \sum_1^n a_i \mathbf{1}_{A_i} \geqslant 0$ *is simple, then* $\sum_{i=1}^{n} a_i \mu(A_i) = \sup\{\int_X t \, d\mu \mid 0 \leqslant t \leqslant s, \text{ simple}\}$.

① $t \leq s \Rightarrow \int_X t \, d\mu \leq \int_X s \, d\mu \Rightarrow \int_X s \, d\mu \geq \sup_{\substack{0 \leq t \leq s \\ t \text{ simple}}} \int_X t \, d\mu$

② Choose $s = t$ & get equality.

Theorem 7.13 (Monotone convergence). *Say* $(f_n) \to f$ *almost everywhere,* $0 \leqslant f_n \leqslant f_{n+1}$*, then* $(\int_X f_n \, d\mu) \to \int_X f \, d\mu$.

Pf: ① $\lim_{n\to\infty} \int_X f_n \, dx$ exists (Yes. $\int f_n \leq \int f_{n+1}$)

(could be ∞)

② $\int_X f_n \, d\mu \leq \int_X f \, d\mu \Rightarrow \lim \int_X f_n \, d\mu \leq \int_X f \, d\mu$.

③ NTS $\liminf \int_X f_n \, d\mu \geq \int f \, d\mu$.

Pf: Let s simple, $0 \leq s \leq f$. enough to show $\lim \int f_n \geq \int s$

Let $E_n = \{ f_n \geq s(1-\varepsilon) \}$ Note $E_n \subseteq E_{n+1}$

Say $\bigcup E_n = X$. ~~$\subset$~~ (oops)

Clearly $\displaystyle\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq (1-\varepsilon)\int_{E_n} s \, d\mu.$

$$= (1-\varepsilon)\sum_{i=1}^{m} a_i \, \mu(A_i \cap E_n) \qquad \left(s = \sum_{i=1}^{m} a_i \mathbb{1}_{A_i} \right)$$

$$\xrightarrow{n \to \infty} (1-\varepsilon)\sum_{i=1}^{m} a_i \, \mu(A_i) = (1-\varepsilon)\int_X s \, d\mu.$$

$\Rightarrow \displaystyle\lim_{n\to\infty} \int_X f_n \, d\mu \geq (1-\varepsilon)\int_X s \, d\mu \quad \forall \, s$ simple, $0 \leq s \leq f$. $\Rightarrow$ QED.

Theorem 7.14. *If f, g are integrable, then $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$.*

Pf: ① Say $f, g \geq 0$. Know $\exists\, s_n, t_n$ simple $\ni$

$$(s_n) \to f, \quad (t_n) \to g, \quad 0 \leq s_n \leq s_{n+1} \quad 0 \leq t_n \leq t_{n+1}$$

M.C. $\Rightarrow$

$$\int_X (f+g)\,d\mu \overset{\text{M.C}}{=} \lim_{n\to\infty} \int_X (s_n + t_n)\,d\mu = \lim_{n\to\infty} \left(\int_X s_n\,d\mu + \int_X t_n\,d\mu\right)$$

linearity for simple fns.

$$\overset{\text{M.C}}{=} \int_X f\,d\mu + \int_X g\,d\mu$$

② Lemma: Say $f = g - h$ where $g \,\&\, h \geq 0$ $(f, g, h \in L^1)$

then $\int_X f\,d\mu = \int_X g\,d\mu - \int_X h\,d\mu$.

Pf: $f = f^+ - f^- = g - h \Rightarrow f^+ + h = f^- + g$ (all ≥ 0)

By ① $\Rightarrow \int_X f^+ + \int_X h = \int_X f^- + \int_X g$

$\Rightarrow \int_X f^+ - \int_X f^- = \int_X h - \int_X g$ QED.

③ NTS $f, g \in L^1$, NTS $\int (f+g) = \int f + \int g$.

Pf: $f+g = (f^+ - f^-) + g^+ - g^- = \underbrace{(f^+ + g^+)}_{\geq 0} - \underbrace{(f^- + g^-)}_{\geq 0}$

By ② $\Rightarrow \int_X (f+g)\,d\mu = \int_X (f^+ + g^+)\,d\mu - \int_X (f^- + g^-)\,d\mu$

$$\overset{\text{by ①}}{=} \int_X f^+ + \int_X g^+ - \int_X f^- - \int_X g^-$$

$$= \left(\int_X f\,d\mu\right) + \left(\int_X g\,d\mu\right)$$

QED.