Definition 6.25. A function $s\colon X \to \mathbb{R}$ is called *simple* if s is measurable, and has finite range (i.e. $s(\mathbb{R}) = \{a_1, \dots a_n\}$).

X

Question 6.26. *Why bother with simple functions?*

Eg: $A \in \Sigma$. $s = \mathbb{1}_A$ $\left(\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases} \right)$

$\left(A \in \Sigma \Rightarrow s \text{ meas } \left(\because \{s < \alpha\} = \begin{cases} X & \alpha > 1 \\ A & \alpha \in (0,1) \leftarrow \in \Sigma \text{ (by ass.)} \\ \emptyset & \alpha \leq 0 \end{cases} \right) \right).$

$A_1, \dots A_n \in \Sigma$. $a_1, \dots a_n \in \mathbb{R}$,

$$s = \sum_1^n a_i \mathbb{1}_{A_i}.$$

$\cancel{\bigotimes}$: $\int_X s \, d\mu \stackrel{\text{def}}{=} \sum a_i \mu(A_i) \leftarrow$

$\forall f$ not simple define $\int f$ by approx f by simple fns.

Theorem 6.27. *If $f \geqslant 0$ is a measurable function, then there exists a sequence of simple functions (s_n) which increases to f.*

Corollary 6.28. *If $f\colon X \to \mathbb{R}$ is measurable, then there exists a sequence of simple functions (s_n) such that $(s_n) \to f$ pointwise, and $|s_n| \leqslant |f|$.* (Dominated).

Pf:

$$f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right) = A_{k,n} \in \Sigma.$$

Try $s_n = \sum_{k=0}^{n^2} \frac{k}{n} \mathbb{1}_{A_{k,n}}$ $\quad$ ($(s_n) \to f$ but need not be inc)

$$f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) = A_{k,n}. \quad \& \text{ let } s_n = \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \mathbb{1}_{A_{k,n}}.$$

s_n simple, $s_{n+1} - s_n \geqslant 0$

& $|s_n(x) - f(x)| \leq \frac{1}{2^n} \Rightarrow (s_n) \to f$ & (s_n) inc

QED.

Pf of Cor: $f^+ = \max\{f, 0\} = f \vee 0$ (meas)

$f^- = -\min\{f, 0\} = -(f \wedge 0)$ (meas)

$f = f^+ - f^-$.

By thm $\exists (s_n)$ simple + $(s_n) \longrightarrow f^+$
& $\exists (t_n)$ simple + $(t_n) \longrightarrow f^-$

then $(s_n - t_n) \longrightarrow f$, $s_n - t_n$ is simple
& $|s_n - t_n| \leq |f|$. QED.

Q: f meas $\not\Rightarrow$ f cts

Q: $\exists$ f meas + f is not cts anywhere?

Yes: $f = \mathbb{1}_{\mathbb{Q}}$.

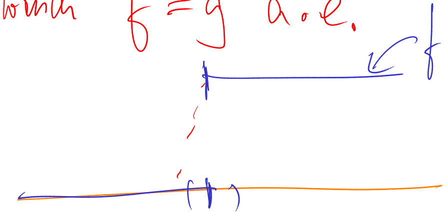
Theorem 6.29 (Lusin). *Let μ be a finite regular measure on a metric space X. Let $f : X \to \mathbb{R}$ be measurable. For any $\varepsilon > 0$ there exists a continuous function $g \colon X \to \mathbb{R}$ such that $\mu\{f \neq g\} < \varepsilon$.*

Cor (D.Lmv): $\exists\, g : X \to \mathbb{R} \ni f = g$ a.e. & f is cts.

(False!)

Cor: f is cts a.e. (FALSE)

$\exists$ meas fns $\ni \nexists$ a cts fn g for which $f = g$ a.e.



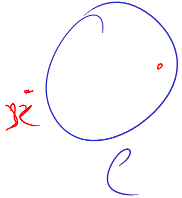
Lemma 6.30 (Tietze's extension theorem). *If $C \subseteq X$ is ~~continuous~~ closed, and $f : C \to \mathbb{R}$ is continuous, then there exist $\bar{f} : X \to \mathbb{R}$ such that $\bar{f} = f$ on C.* AND f is cts

Rem: If X is a top space the the pf is hard.

Pf:

$$\bar{f}(x) = \begin{cases} \inf\left\{ c \in C \;\middle|\; f(c) + \dfrac{d(x,c)}{d(x,C)} - 1 \right\} & x \notin C \\ f(x) & x \in C \end{cases}$$

ε–δ check $\bar{f}$ is cts.



Lemma 6.31. *Let $f: X \to \mathbb{R}$ be measurable. For every $\varepsilon > 0$, there exists $C \subseteq X$ closed such that $\mu(X - C) < \varepsilon$ and $f: C \to \mathbb{R}$ is continuous.*

Pf: Case I: $f : X \to [0,1]$ (f is bdd)

disj $\to$ $A_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \in \Sigma \Rightarrow \exists K_{n,k}$ cpt s.t. $K_{n,k} \subset A_{n,k}$

s.t. $\mu(A_{n,k} - K_{n,k}) < \frac{\varepsilon}{\cancel{2^n}\, 4^n}$

Let $C_n = \bigcup_k K_{n,k}$. Note $\mu(C_n^c) \leq 2^n \cdot \frac{\varepsilon}{\cancel{2^n}\, 4^n} = \frac{\varepsilon}{2^n}$

Let $C = \bigcap_1^\infty C_n$. Note C is closed & $\mu(C^c) \leq \sum \frac{\varepsilon}{2^n} = \varepsilon$

m

Claim: f is cts on C.

Pf: Let $s_n = \sum_{k=0}^{2^n} \frac{k}{2^n} \mathbb{1}_{K_{n,k}}$

Note s_n is cts on $\bigcup_k K_{n,k} = C_n \Rightarrow s_n$ cts on C

($K_{n,k}$ are disj as k varies).

Note $|s_{n+1} - s_n| \leq \frac{1}{2^{n+1}}$ on $C \Rightarrow (s_n) \to f$ unif on C

$\Rightarrow f$ is cts on C

QED.

Case II: f not bdd

Set $g = \tan^{-1}(f)$

Proof of Lusin's theorem. Previous two lemmas. $\longrightarrow$ $\exists\, C$ closed & $f : C \longrightarrow \mathbb{R}$ is cts □

& $\mu(X - C) < \varepsilon$.

Use Tiezte to extend f.

Proof of Lemma 6.31. □