Definition 6.13 (Cantor function)**.** Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0,x])/H_\alpha(C)$.

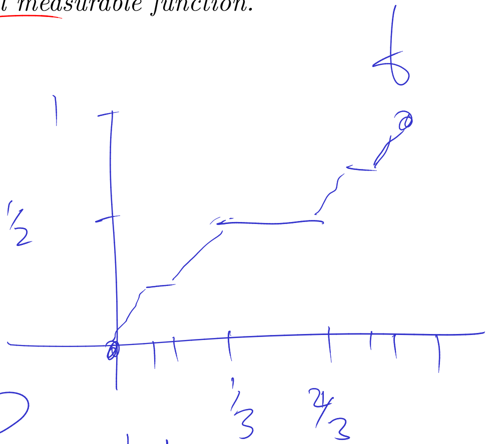
(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C^c, and $f' = 0$ wherever defined.)
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)
(3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$).

Proposition 6.14. *The function $g\colon [0,1] \to C$ is a strictly injective Borel measurable function.*

Pf f is cts:

$$f(x) - f(x - 1/n) = \frac{H_\alpha\left((x - \frac{1}{n}, x] \cap C\right)}{H_\alpha(C)} \xrightarrow{n \to \infty} \frac{H_\alpha(\{x\} \cap C)}{H_\alpha(C)} = 0$$

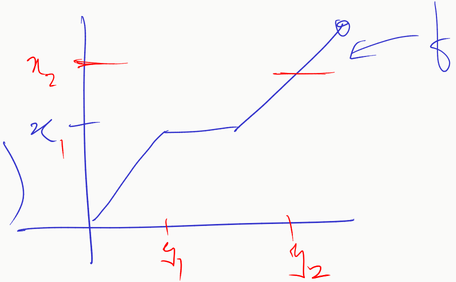
$\Rightarrow f$ cts.



$g = \underline{f}^{-1}$: $\underline{g(x)} = \inf\{y \mid f(y) = x\}$

(Q: $\{y \mid f(y) = x\} \neq \phi$? $(x \in [0,1])$

A: Yes $\rightarrow$ int val thm)



f cts $\Rightarrow \inf\{y \mid f(y) = x\} = \min\{y \mid f(y) = x\}$

$\Rightarrow f(g(x)) = x$

Claim 1: g is <u>strictly</u> inc. ✓

Claim 2: g is Borel meas ($\because \{g < \alpha\}$ is an interval $\forall \alpha$)

Claim 3: Range $(g) \subseteq C$

Theorem 6.15. $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$.

Pf: Let $A \subseteq [0,1]$ be non meas

Q: $g(A) \rightarrow$ meas? Yes: $g(A) \subseteq C$ null $\Rightarrow g(A) \in \mathcal{L}(\mathbb{R})$

Q2: Is $g(A) \in \mathcal{B}(\mathbb{R})$?

NO! If $g(A) \in \mathcal{B} \Rightarrow \underbrace{g^{-1}(g(A))}_{A} \in \mathcal{B}(\mathbb{R})$ ($\because g$ is meas)

But $A \notin \mathcal{L}(\mathbb{R})$ by const. Contradiction

QED.

Theorem 6.16. *There exists $h_1, h_2 \colon \mathbb{R} \to \mathbb{R}$ such that h_1 is $\mathcal{L}(\mathbb{R})$-measurable, h_2 is $\mathcal{B}(\mathbb{R})$ measurable, but $h_1 \circ h_2$ is not $\mathcal{L}(\mathbb{R})$ measurable.*

Remark 6.17. Note $h_2 \circ h_1$ has to be $\overset{\mathcal{L}}{\cancel{\mathcal{B}}}(\mathbb{R})$-measurable.

$E \subseteq X$,

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Pf: $A \subseteq [0,1]$, $A \notin \mathcal{L}(\mathbb{R})$

$g(A) \in \mathcal{L}(\mathbb{R})$

Let $h_1 = \mathbb{1}_{g(A)}$ (h_1 is $\mathcal{L}$-meas)

Let $h_2 = g$ (h_2 is $\mathcal{B}$ meas)

Note $h_1 \circ h_2 = \mathbb{1}_{g(A)} \circ g = \begin{cases} 1 & \text{on } A \\ 0 & \text{on } A^c \end{cases} = \mathbb{1}_A$ not $\mathcal{L}(\mathbb{R})$ meas

QED.

a.e.

Definition 6.18. Let (X, Σ, μ) be a measure space. We say a property P holds almost everywhere if there exists a null set N such that P holds on N^c.

Example 6.19. If f, g are two functions, we say $f = g$ almost everywhere if $\{f \neq g\}$ is a null set.

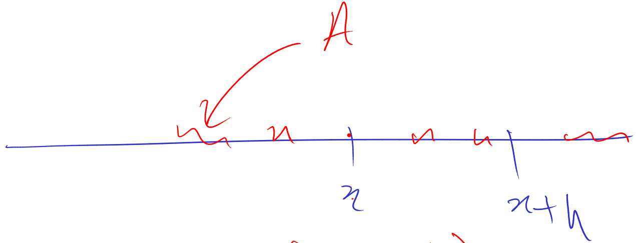
Example 6.20. Almost every real number is irrational.

(IOU Pf)

Example 6.21. If $A \in \mathcal{L}(\mathbb{R})$, then $\displaystyle\lim_{h \to 0} \frac{\lambda(A \cap (x, x+h))}{h} = \mathbf{1}_A(x)$ for almost every x. (Contrast with HW3, Q3b)

Example 6.22. Let $x \in (0, 1)$, and p_n/q_n be the n^{th} convergent in the continued fraction expansion of x. Then $\displaystyle\lim_{n \to \infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$.

$A = [0,1]$.



$$\mathbb{1}_A \overset{a.e}{=} \mathbb{1}_{[0,1)} = \lim_{h \to 0} \frac{\lambda(A \cap (x, x+h))}{h}$$

$$\left(\Rightarrow \nexists E \subseteq \mathbb{R} \text{ meas } \ni \forall (a,b),\ \frac{\lambda(E \cap (a,b))}{b-a} \in [k, 1-k] \right)$$

$x \in [0,1] \longrightarrow$ cont fraction for x

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Truncate to n terms. $\dfrac{p_n(x)}{q_n(x)} = n^{th}$ conv of the x

$$\lim_{n\to\infty} \frac{p_n(x)}{q_n(x)} = x \quad \forall x$$

Expect $q_n(x) \longrightarrow \infty$. Q: How fast?

$$\lim_{n\to\infty} \frac{\ln q_n(x)}{n} = \frac{\pi^2}{12 \ln 2} \quad \cancel{\forall}\ x$$

on almost every!

Assume hereafter (X, Σ, μ) is complete.

Proposition 6.23. *If $f = g$ almost everywhere and f is measurable, then so is g.*

Pf: NTS g meas. Let $N = \{f \neq g\}$ (null)

Pick $U \subseteq \mathbb{R}$ open.

$$g^{-1}(U) = \left(g^{-1}(U) \cap N^c\right) \cup \left(g^{-1}(U) \cap N\right)$$

$$= \underbrace{\left(f^{-1}(U) \cap N^c\right)}_{\in \Sigma} \cup \underbrace{\left(g^{-1}(U) \cap \underline{N}\right)}_{\in \Sigma} \Rightarrow \text{QED}.$$

Proposition 6.24. *If $(f_n) \to f$ almost everywhere, and each f_n is measurable, then so is f.*

Pf: $N^c = \{x \mid \lim_{n\to\infty} f_n(x) = f(x)\}$ N is null, ($\Rightarrow$ meas).

$$\underbrace{\mathbb{1}_{N^c} f = \lim_{n\to\infty} \underbrace{\mathbb{1}_{N^c} f_n}_{\text{meas}}}_{\text{meas (last time)}} \quad (\forall x)$$

$\Rightarrow \mathbb{1}_{N^c} f$ is meas

$\mathbb{1}_{N^c} f = f$ a.e. $\Rightarrow f$ meas

QED

~~Claim~~ $\boxed{\text{HW 3b}}$ Claim $\nexists E \subseteq \mathbb{R}$ s.t. $\forall$ intervals I, $\dfrac{\lambda(E \cap I)}{\lambda(I)} \in [k, 1-k]$; $(k > 0)$

$$\Lambda = \{ A \in \mathcal{B} \mid k\lambda(A) \le \lambda(A \cap E) \le (1-k)\lambda(A) \}$$

① $\cancel{\mathbb{R}}$ $[0,1] \in \Lambda$

② $A \subseteq B$, $A, B \in \Lambda$, NTS $B - A \in \Lambda$

$\lambda($

$k = .2$