Theorem 6.5. *Say $f: X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 6.6. *Let $f: X \to Y$ be arbitrary, and Σ be a σ-algebra on X. Then $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$ is a σ-algebra (on Y).*

f – meas means $\forall\, U \subseteq Y$ open $\Rightarrow f^{-1}(U) \in \Sigma$.

($f: X \to Y$

Σ – σ-alg on X)

Say τ is σ-alg on Y. $f: X \to Y$

Then $f^{-1}(\tau)$ is a σ-alg on X (True → not useful)

(X, Σ) $f: X \to Y$. $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$

Σ' is a σ-alg.

If f meas $\Rightarrow \Sigma' \supseteq$ open sets $\Rightarrow \Sigma' \supseteq \mathcal{B}(Y)$ PED Thm

Corollary 6.7. *Let* $f\colon X \to [-\infty, \infty]$. *Then* f *is measurable if and only if for all* $a \in \mathbb{R}$, *we have* $\{f < a\} \in \Sigma$.

Lemma 6.8. *If $f\colon X \to \mathbb{R}^m$ is measurable, and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f\colon X \to \mathbb{R}^n$ is measurable.*

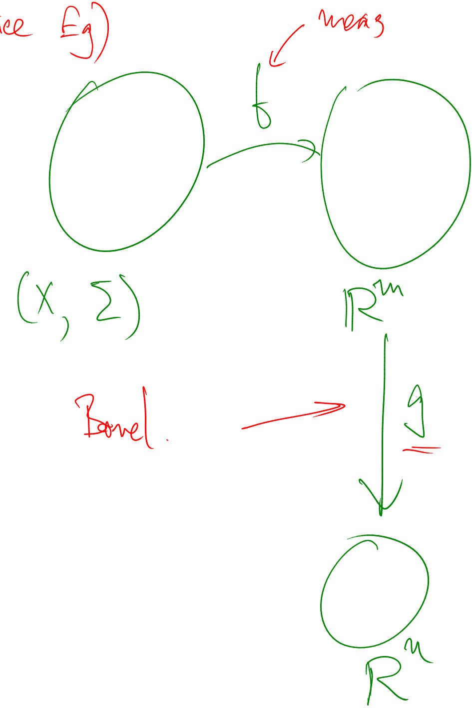
Question 6.9. *Is the above true if g was Lebesgue measurable?* (false IOU nice Eg)

Pf: $(g\circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)})$

$= f^{-1}(\text{Borel})$ $\quad (\because g \to \text{Borel})$

$\overset{\text{Thm}}{=} \in \Sigma$

QED.



Theorem 6.10. *Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. Then* $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ *and* $\lim f_n$ *(if it exists) are all measurable.*

Q: $(f_n) \to f$ ptwise. f_n is R-int. Q: Is f R-int? (No)

Pf of thm: ① $\sup f_n$ meas: $\{\sup f_n < \alpha\} = \bigcap_{n=1}^{\infty} \{f_n < \alpha\}$

$(\{f<\alpha\} = \{x \mid f(x) < \alpha\})$

f_n meas $\forall n \Rightarrow \bigcap_{n=1}^{\infty} \{f_n \le \alpha\} \in \Sigma \Rightarrow \{\sup_n f_n < \alpha\} \in \Sigma$

By Lemma $\Rightarrow$ QED.

⑤ Let $f(x) = \begin{cases} \lim f_n(x) & \text{if the lim exists} \\ 0 & \text{otherwise.} \end{cases}$

Q: Is f meas. (each f_n is meas)

$E = \{ \underbrace{\limsup f_n} = \underbrace{\liminf f_n} \} \in \Sigma$ (check directly)

$\left(\mathbb{1}_E = \text{indicator fn of } E \right.$

$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$

$$f(x) = \lim_{n\to\infty} \underbrace{\mathbb{1}_E f_n(x)}$$

$$= \liminf_{n\to\infty} \underline{\underline{\mathbb{1}_E f_n}}$$

(check $\mathbb{1}_E f_n$ is meas $\forall n$)

$\Rightarrow f$ is meas

QED.

Theorem 6.11. *Let $f, g\colon X \to \mathbb{R}$. The function $(f, g)\colon X \to \mathbb{R}^2$ is measurable if and only if both f and g are measurable.*

$F = (f, g)$ $\quad F : X \to \mathbb{R}^2$ $\quad F(x) = (f(x), g(x))$

Pf: (1) Say f, g are meas. $\forall\, U, V \subseteq \mathbb{R}$, U, V open $\Rightarrow f^{-1}(U)$ & $g^{-1}(V) \in \Sigma$

$$F^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \in \Sigma.$$

Let $\Sigma' = \{E \subseteq \mathbb{R}^2 \mid F^{-1}(E) \in \Sigma\}$. Know Σ' is a σ-alg

Know $\Sigma' \supseteq \{U \times V \mid U, V \subseteq \mathbb{R} \text{ open}\}$ $\Rightarrow \Sigma' \supseteq \mathcal{B}(\mathbb{R}^2)$ QED

(2) Conversely: F meas. $\pi_1(x, y) = x$ cts fn ($\Rightarrow$ Borel)

$f(x) = \pi_1 \circ F(x)$ $\Rightarrow f$ is meas (Borel composed with meas) QED

Corollary 6.12. *If $f, g\colon X \to \mathbb{R}$ are measurable, then so is $f + g$, fg and f/g (when defined).*

Pf: ① $f+g$: $\quad F(x) = (f(x), g(x))$. $\quad G(x, y) = x + y$

$$\left.\begin{array}{l} f(x) + g(x) = \underbrace{G}_{\substack{\text{cts} \\ \text{(Borel)}}} \circ \underbrace{F}_{\text{meas}}(x) \end{array}\right\} \Rightarrow \text{meas (by lemma)}$$

QED.

(Devils staircase)

Definition 6.13 (Cantor function)**.** Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0,x])/H_\alpha(C)$.

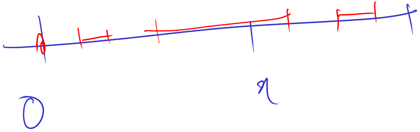
(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C^c, and $f' = 0$ wherever defined.)

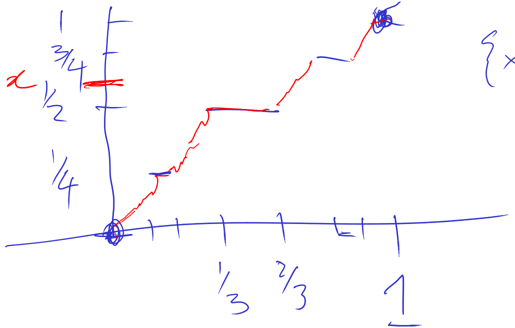
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)

(3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$). $\leftarrow f(g(x)) = x$

Proposition 6.14. *The function $g\colon [0,1] \to C$ is a strictly injective Borel measurable function.*

$f(x) =$





$\{x \mid g(x) \leq \alpha\}$

$= \{x \mid x \leq f(\alpha)\}$

interval.

$g(x) =$