Corollary 5.16. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 5.17. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\Sigma_\mu \stackrel{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$.

Definition 5.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Theorem 5.19. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

What time $\bar{\mu}$ → well def. ① Σ_μ is a σ-alg. Pf: ⓐ $X \in \Sigma_\mu$.

ⓑ If $A \in \Sigma_\mu \Rightarrow A^c \in \Sigma_\mu$ (Pf: Find $F \subseteq A \subseteq G \Rightarrow G^c \subseteq A^c \subseteq F^c$)

ⓒ $A_i \in \Sigma_\mu$. NTS $\bigcup_i^\infty A_i \in \Sigma_\mu$ (Pf: $\forall i$ $\exists F_i \subseteq A_i \subseteq G_i$ + $\mu(G_i - F_i) = 0$

set $F = \bigcup F_i$, $G = \bigcup G_i$) ✓

② $\bar{\mu}$ a meas: $A_i \in \Sigma_\mu$, disj. Find F_i, G_i + $F_i \subseteq A_i \subseteq G_i$ & $\mu(G_i - F_i) = 0$

F_i disj $\quad \bigcup F_i \subseteq \bigcup A_i \subseteq \bigcup G_i$ & $\mu(\bigcup G_i - \bigcup F_i) = 0 \Rightarrow \bar{\mu}(\bigcup A_i) = \mu(\bigcup F_i) = \sum \bar{\mu}(A_i)$ QED

Theorem 5.20. Σ_μ *is the smallest μ-complete σ-algebra containing* Σ.

Corollary 5.21. $\Sigma_\mu = \sigma(\Sigma \cup \mathcal{N})$.

$\boxed{\textbf{Corollary 5.22. } \mathcal{L}(\mathbb{R}^d) = \sigma(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N}).}$

Pf of 5.20: Say (X, τ, ν) is a meas space. $\tau \supseteq \Sigma$ & ν extends μ.

τ - complete $\Rightarrow \tau \supseteq \Sigma_\mu$.

Pf: $A \in \Sigma_\mu$. find $F, G \in \Sigma$ $\ni$ $F \subseteq A \subseteq G$ & $\mu(G - F) = 0$

$\Rightarrow \nu(G - F) = 0$

$\Rightarrow A - F$ is null $\left(A - F \subseteq G - F \longrightarrow \begin{array}{l} \mu \text{ null} \\ \nu \text{ null} \end{array} \right)$

$A - F \in \tau \Rightarrow A \in \tau$ QED.

Remark 5.23. There could exist μ-null sets that are not in Σ.

Say $\lambda \to$ lebesgue meas on $[0,1]$.

$\to \Sigma = \{\phi, [0,1]\}$.

Q: Null sets of (Σ, λ): $N = \{\phi\}$, $\cancel{\{0\}}$

6. Measurable Functions

Definition 6.1. Let (X, Σ, μ) be a measurable space, and (Y, τ) a topological space. We say $f\colon X \to Y$ is measurable if $f^{-1}(\tau) \subseteq \Sigma$.

$\forall U \in \tau,\ f^{-1}(U) \in \Sigma$

Remark 6.2. Y is typically $[-\infty, \infty]$, $\mathbb{R}^d$, or some linear space.

Remark 6.3. Any continuous function is Borel measurable, but not conversely.

Question 6.4. *Say $f\colon X \to Y$ is measurable. For every $B \in \mathcal{B}(Y)$, must $f^{-1}(B) \in \Sigma$?*

$\forall U \subseteq Y$ open, $f^{-1}(U) \in \Sigma$

$$(X, \Sigma) \xrightarrow{f} (Y, \tau)$$

meas sp $\qquad$ top space

Q: $\forall B \subseteq Y$ Borel, is $f^{-1}(B) \in \Sigma$?

Q2: $Y = \mathbb{R}$. $f: X \to Y$ is meas.

Q: $\forall B \in \mathcal{L}(\mathbb{R})$ is $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$?

$\boxed{\text{(NO!)}}$

Theorem 6.5. *Say $f\colon X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 6.6. *Let $f\colon X \to Y$ be arbitrary. Then $\Sigma' = \cancel{\{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}}$ is a σ-algebra (on $\cancel{Y}$ X).*

$\Sigma' = \{f^{-1}(B) \mid B \in \Sigma\}$

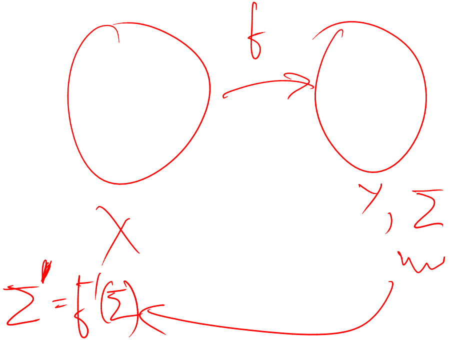
$$\Sigma' = f^{-1}(\Sigma) \qquad (\Sigma \text{ is a } \sigma\text{-alg on } Y)$$

Pf: $A_i \in f^{-1}(\Sigma) = \Sigma'$

Write $A_i = f^{-1}(B_i)$, $B_i \in \Sigma$

$$f^{-1}\left(\bigcup B_i\right) = \bigcup f^{-1}(B_i) = \bigcup A_i$$

$\Rightarrow \bigcup A_i \in \Sigma'$. $\Rightarrow$ lemma



Pf of thm: $f^{-1}(\mathcal{B}(Y))$ is a σ-alg. f meas $\Rightarrow \forall\, U \subseteq Y$ open $\Rightarrow f^{-1}(U) \in \Sigma$.

$$\Rightarrow f^{-1}(\mathcal{B}(Y)) \subseteq \Sigma.$$

Corollary 6.7. *Let $f\colon X \to [-\infty, \infty]$. Then f is measurable if and only if for all $a \in \mathbb{R}$, we have* $\boxed{\{f < a\}} \in \Sigma$.

(X, Σ) meas space.

$\{f<a\} = \{x \in X \mid f(x) < a\}$

$= f^{-1}([-\infty, a))$

Pf: any open set can be expressed as a ct[ble] union of intervals.

& $f^{-1}(a,b) = \{f<b\} \cap \underbrace{\{f>a\}}$ $\in \Sigma$

$\{f<b\} \in \Sigma$.

$\{f>a\} = \left(\bigcup_{n=1}^{\infty} \underbrace{\{f \geq a+\tfrac{1}{n}\}}_{\{f<a+\frac{1}{n}\}^c}\right) \in \Sigma$

Lemma 6.8. *If $f\colon X \to \mathbb{R}^m$ is measurable, and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f\colon X \to \mathbb{R}^n$ is measurable.*

Question 6.9. *Is the above true if g was Lebesgue measurable?* (False).

Pf Lemma: $U \subseteq \mathbb{R}^n$

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

$$g^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)$$

(By lemma) $\Rightarrow f^{-1}(g^{-1}(U)) \in \Sigma.$

$$\begin{tikzcd} (X,\Sigma) \arrow[r, "f"] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^n \end{tikzcd}$$

g is $\boxed{\text{Borel}}$ meas

$(g^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)\ \forall U$ open$)$.