HW Q 3/4:

Hausdorff meas: $A \subseteq \bigcup_{i}^{\infty} E_i$, $\operatorname{diam}(E_i) < \delta$

$$\lim_{\delta \to 0} H_{\alpha,\delta}(A) = H_\alpha(A) \leftarrow H_{\alpha,\delta}(A) = \inf \Big\{ \sum c_\alpha \operatorname{diam}(E_i)^\alpha \;\Big|\; \Big\}$$

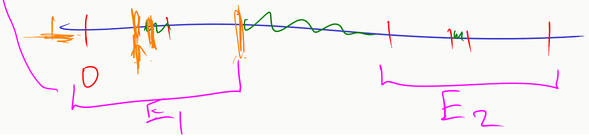
Sph meas: $$S_{\alpha,\delta}(A) = \inf \Big\{ \sum c_\alpha \operatorname{diam}\big(B(x_i, r_i)\big)^\alpha \;\Big|\; A \subseteq \bigcup_{i}^{\infty} B(x_i, r_i) \ \& \ 2r_i \le \delta \Big\}$$

$$S_\alpha \overset{?}{=} H_\alpha \Rightarrow H_\alpha \le S_\alpha \qquad S_\alpha \le 2^d H_\alpha$$

Claim: $\boxed{H_\alpha \neq S_\alpha}$ in general

Eg: Cantor Set.

$\delta = 1/3$ cover C by (1) sets of diam 1/3, (2) balls of diam 1/3



Is E_1 a Ball in the Cantor set (NO)

5.3. **Non-measurable sets.**

Theorem 5.13. *There exists* $\boxed{E \subseteq \mathbb{R} \text{ such that } E \notin \mathcal{L}(R).}$

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)
(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.
(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.
(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.
(5) $\boxed{\lambda(E) > 0}$

(axiom of choice).

$\forall \alpha \in \mathbb{R},\ E \cap C_\alpha = \{e_\alpha\}$.

Claim $E \notin \mathcal{L}(R)$.

Pf: By contradiction

Pf of (5):

$$\bigcup_{q \in \mathbb{Q}} E + q = \mathbb{R}$$

ctable disjt union

∞ meas.

$$\lambda(q + E) = \lambda(E) \ \forall q$$

$\Rightarrow \lambda(E) > 0$

QED.

(6) $\lambda(E) = 0$ (contradiction).

Let $n \in \mathbb{N}$. $E_n = E \cap (-n, n)$. $\lambda(E) = \lim_{n \to \infty} \lambda(E_n)$

Claim $\lambda(E_n) = 0 \quad \forall n$. $\left(\Rightarrow \lambda(E) = 0 \Rightarrow \text{QED} \right)$.

Pf of Claim: Let $A = \bigcup_{\substack{|q| \leq 1 \\ q \in \mathbb{Q}}} (q + E_n) \longleftarrow$ ctable disj union.

A is bdd $\Rightarrow \lambda(A) < \infty$ & $\lambda(q + E_n) = \lambda(E_n) \quad \forall q$.

$\Rightarrow \lambda(A) = \lambda(E_n) = 0$ QED.

Theorem 5.14. *Let $A \subseteq \mathbb{R}^d$. Every subset of A is Lebesgue measurable if and only if $\lambda(A^*) = 0$.*

Proof. One direction is immediate. The other direction is accessible with what we know so far, but we won't do the proof in the interest of time. $\square$

Thm: $\exists\, A \subseteq \mathbb{R}$ $\ni$ $E \subseteq A,\ E \in \mathcal{L}(\mathbb{R}) \Rightarrow \lambda(E) = 0$

and $E \subseteq A^c,\ E \in \mathcal{L}(\mathbb{R}) \Rightarrow \lambda(E) = 0$

(IOU: $\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d)$

~~Easy~~ Easier when $d \geq 2$ $\longrightarrow$ on HW)

5.4. Completion of measures.

Theorem 5.15. *$A \in \mathcal{L}(\mathbb{R}^d)$ if and only if there exist $F, G \in \mathcal{B}(\mathbb{R}^d)$ such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$.*

Pf: $\forall n \in \mathbb{N}$, $\exists$ U_n open, C_n closed $\ni$ $C_n \subseteq A \subseteq U_n$ & $\lambda(U_n - C_n) < \frac{1}{n}$

Let $F = \bigcup C_n$ (F_σ)

$G = \bigcap U_n$ (G_δ).

Clearly $F \subseteq A \subseteq G$ & $\lambda(G - F) \leq \lambda(C_n - U_n) \leq \frac{1}{n}$ $\forall n$

$\Rightarrow \lambda(G - F) = 0$ QED.

Corollary 5.16. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 5.17. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\Sigma_\mu \stackrel{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$. $= \mu(G)$ (IOU Thm well defined)

Definition 5.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Done

Theorem 5.19. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

$(\mathbb{R}^d, \mathcal{L}, \lambda)$ is comp

(IOU) $\Rightarrow (\mathbb{R}^d, \mathcal{B}, \lambda)$ is not

Check $\bar{\mu}$ is well defined:

Say $A \in \Sigma_\mu$ $\quad \exists\, F_1 \subseteq A \subseteq G_1$, $F_2 \subseteq F \subseteq G_2$ $\quad F_i, G_i \in \Sigma$ & $\mu(G_i - F_i) = 0$

$\Rightarrow F_1 \subseteq F_1 \cup F_2 \subseteq A \subseteq G_1 \cap G_2 \subseteq G_1$

$\mu(G_1 - F_1) = 0 \Rightarrow \mu((F_1 \cup F_2) - F_1) = 0 \Rightarrow \mu(F_2) = \mu(F_1) \Rightarrow \bar{\mu}$ well def

QED.