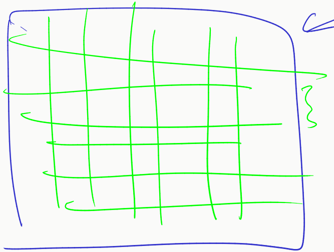
Q2c

$\alpha = d$. NTS $H_d(B) < \infty$ if B is bdd.

$B = [0,1]^2$.



$I_k = \{$ cell of side len $\frac{1}{n}$ & bot left corn $(\frac{l}{n}, \frac{m}{n})$ $\forall\, l, m \in \{0, \dots n\} \}$

$H_d = \lim_{\delta \to 0} \underbrace{H_{d,\delta}(B)}$ ← cover B with sets of diam $\leq \delta$.

Claim: $\forall \delta,$ ~~$\forall \delta$, n~~ $\exists C > 0$ s.t. $\forall \delta$, $H_{d,\delta}(B) \leq C$

Q2 b) NTS σ-alg $\supseteq \mathcal{B}(\mathbb{R}^d)$

① $\forall \delta > 0$, $\Sigma_\delta = \sigma$ alg from Carathéodory for $H_{\alpha,\delta}$

Claim $\Sigma_\delta \supseteq$ open sets

Claim ② $H_{\alpha,\delta}(A) = H_{\alpha,\delta}(A \cap E) + H_{\alpha,\delta}(A \cap E^c)$ (say true for some open set E & all $\delta > 0$)

$I_k \longrightarrow$ cell of side length $\frac{1}{n}$

$\operatorname{diam}(I_k) \approx \frac{\sqrt{2}}{n}$ $(d=2)$ Want $\frac{\sqrt{2}}{n} < \delta$.

$$H_{2,\delta}(B) \leq \sum_k \rho\left(\operatorname{diam}(I_k)\right) =$$

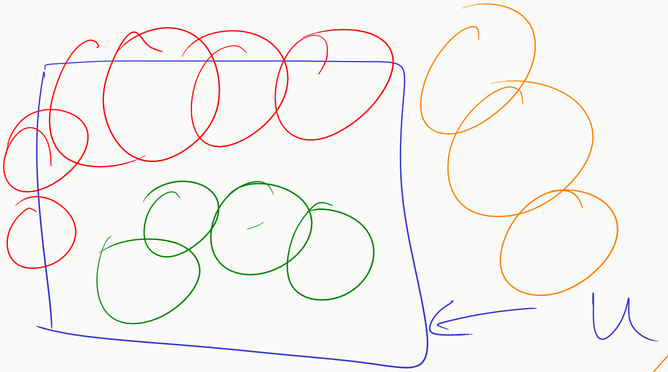
$$n^2 \left(C \frac{\sqrt{2}}{n}\right)^2 \leq 2C \quad \forall \delta.$$

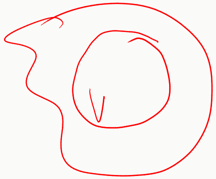
$\sim$

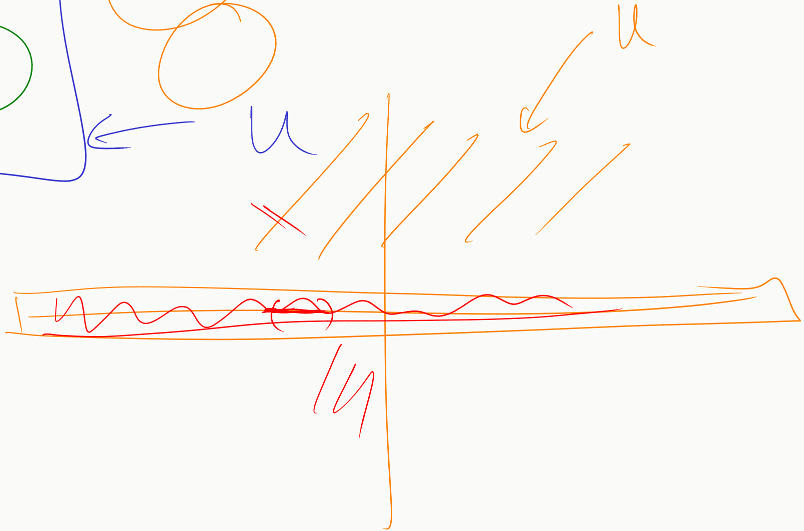
$d = 2$

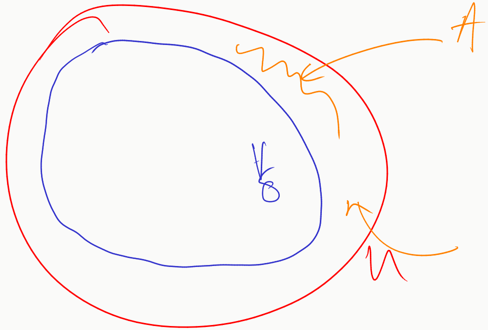
$\to U = $ Half space

$\to \alpha = 1$









1a: NTS $\lambda(TA) = |\det T|\, \lambda(A)$ $\forall A \in \mathcal{L}$.

① Show this $\forall A \to$ cells.

②

Q: $T : \mathbb{R}^d \to \mathbb{R}^d$ linear (inv)

① $A \subseteq \mathbb{R}^d$. $\boxed{A \in \mathcal{L}(\mathbb{R}^d) \iff T(A) \in \mathcal{L}(\mathbb{R}^d)}$

② $\boxed{A = \text{cell} : \lambda(T(A)) = |\det T| \, \lambda(A).} \leftarrow$

③ $\mu(A) = \dfrac{1}{|\det T|} \lambda(T(A))$. $\mu = \lambda$ on cells

$\Rightarrow \mu = \lambda \ \forall A$

$\Sigma = \{ U \in \cancel{\mathcal{L}(\mathbb{R}^d)}\,\mathcal{B} \mid T(U) \in \cancel{\mathcal{L}}\,\mathcal{B}(\mathbb{R}^d) \} \leftarrow \sigma$ alg

$\Sigma \supseteq$ all open sets $\Rightarrow \Sigma \supseteq$ Borel.

2 d:

$$H_\alpha(A) \in (0, \infty) \Rightarrow \forall \beta > \alpha, \ H_\alpha(A) = +\infty$$

Say: $\sum a_i^\alpha \in (0, \infty)$ $\quad (\cancel{\alpha}\, a_i > 0)$

Q: $\sum a_i^\beta$ $\quad$ for $\beta > \alpha$

& $\beta < \alpha$