5.2. **Regularity of measures.**

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$.
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$.

Motivation:

- ▷ Approximation of measurable functions by continuous functions
- ▷ Differentiation of measures
- ▷ Uniqueness in the Riesz representation theorem

Question 5.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?*

(False $\rightarrow$ find a c-ex)

satisfies (1) above. $\Rightarrow \mu$ is regular &

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is ~~regular~~, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\boxed{\mu(U - C) < \varepsilon}$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. □

Theorem 5.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. $\boxed{\text{Then } \mu \text{ is regular}}$. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ ~~closed~~ cpt such that $\mu(U - K) < \varepsilon$.*

$(A\mu = \lambda)$

$\exists$ K cpt & U open s.t.

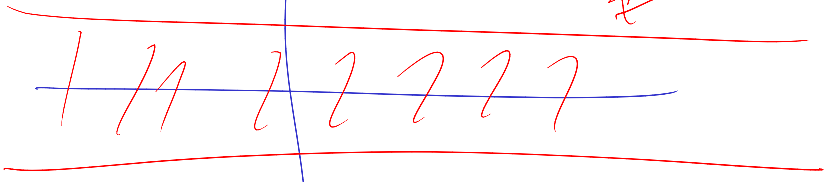
$K \subseteq A \subseteq U$

& $\mu(U - K) < \varepsilon$

$\mu(A) = \sup_{K \subseteq A} \mu(K)$ (K cpt)

A

$\not\!\!\!\!$



$\mathbb{R}^2$

$\forall \varepsilon > 0,\ A \in \mathcal{B}(X)\ \exists\ U \supseteq A,\ U$ open
$K \subseteq A,\ K$ cpt $\Big\}$ $\mu(U - K) < \varepsilon$.

Proof:

(1) Let $\Lambda = \{A \in \mathcal{B}(X) \mid \forall \varepsilon > 0,\ \exists K \subseteq A$ compact, $U \supseteq A$ open, such that $\mu(U - K) < \varepsilon\}$.
(2) Λ contains all open sets.

Let $U \subseteq X$ open. NTS $\forall \varepsilon > 0\ \exists K \subseteq U$ cpt s.t. $\mu(U - K) < \varepsilon$.

Write $U = \bigcup_{1}^{\infty} K_n$, $K_n \subseteq X$ is cpt & $K_n \subseteq K_{n+1}$

(Eg $K_n = \{x \in X \mid d(x, U^c) \geq \frac{1}{n}\}$)

$\Rightarrow \mu(U) = \lim_{n \to \infty} \mu(K_n)$ QED

(3) Λ is a λ-system. (In this case it's easy to directly show that Λ is a σ-algebra.)
(4) Dynkin's Lemma implies $\Lambda \supseteq \mathcal{B}(X)$, finishing the proof.

Pf of (3): (1) $X \in \Lambda$ ($\because X$ is open).

NTS: (2) $A_1, A_2 \in \Lambda$, $A_1 \subseteq A_2 \Rightarrow A_2 - A_1 \in \Lambda$

Pf: Pick $\varepsilon > 0$, $\exists K_i, U_i$ + K_i cpt, U_i open $K_i \subseteq A_i \subseteq U_i$

& $\mu(U_i - K_i) < \varepsilon$ ($\because A_i \in \Lambda$)

$K_2 - U_1 \subseteq A_2 - A_1 \subseteq \underbrace{U_2 - K_1}_{\text{open}}$, & $\mu((U_2 - K_1) - (K_2 - U_1))$

$\leq \mu(U_2 - K_2) + \mu(U_1 - K_1) = 2\varepsilon$

QED

(3) $A_1, A_2 \dots \in \Lambda$. NTS $\bigcup_1^\infty A_i \in \Lambda$.

$A_i \subseteq A_{i+1}$. Pf: $B_i = A_i - A_{i-1}$. Then $\bigcup A_i = \bigcup_{\text{disj}} B_i$

$\forall i,\ B_i \in \Lambda \Rightarrow \exists K_i$ cpt & U_i open s.t. $K_i \subseteq B_i \subseteq U_i$

& $\mu(U_i - K_i) < \frac{\varepsilon}{2^i}$

Certainly $\underbrace{\bigcup_i^{\infty} K_i}_{\text{not cpt}} \subseteq \bigcup A_i \subseteq \underbrace{\bigcup_i^{\infty} U_i}_{\text{open}}$ & $\mu\left(\bigcup U_i - \bigcup K_i\right) \leq \sum \mu_i(U_i - K_i) < \varepsilon$.

take $\bigcup_i^{N}$ for some large N & finish.

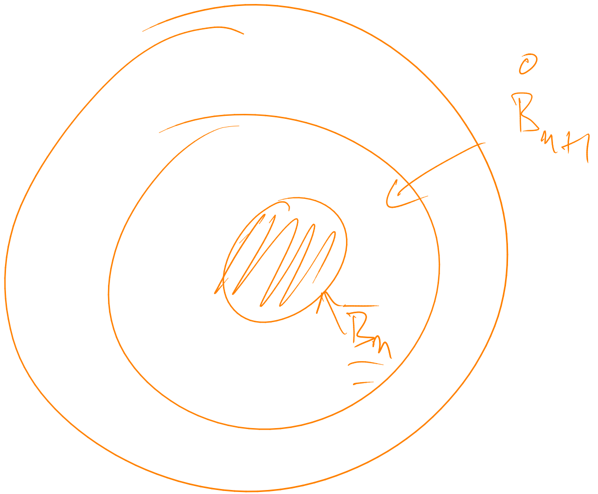
Corollary 5.11. *Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subset X$ such that $\bar{B}_n \subset \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\boxed{\mu(B_n) < \infty}$. Then μ is regular. Further:*

(1) *For any Borel set A, $\mu(A) = \sup\{\mu(K) \mid K \subseteq K$ is compact$\}$.*
(2) *For any $\varepsilon > 0$, there exists $U \supseteq A$ open and $C \subseteq A$ closed such that $\mu(U - C) < \varepsilon$.*

Proof. On homework. $\square$

$X = \bigcup_{1}^{\infty} B_n$

$(\text{Eg}: X = \mathbb{R}^d. \quad B_n = B(0, n))$



Theorem 5.12. *Let $A \in \mathcal{L}(\mathbb{R}^d)$, $\lambda(A)$.* (λ = Lebesgue meas).

(1) $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\} = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ compact}\}$.

(2) *There exists $\varepsilon > 0$, $C \subseteq A$ closed and $U \supseteq A$ open such that $\lambda(U - C) < \varepsilon$.*

Pf of (1): $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\}$ ✓

$\left(\because\ \lambda(A) = \lambda^*(A) = \inf\{\sum \ell(I_k) \mid \bigcup I_k \supseteq A,\ I_k \text{ open cells}\}\right.$

$= \inf\{\lambda(\bigcup I_k) \mid \quad " \quad\quad " \quad\}$ QED)

NTS $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ cpt}\}$.

↳ Case 1: A bdd. $\lambda(A) = \lambda(I) - \lambda(I - A)$, $I \supseteq A$ is some closed cell

& use $\lambda(I - A) = \inf\limits_{U \supseteq I - A} \lambda(U)$.

$\Rightarrow$ (1)

Case 2: $A = \bigcup\limits_{n=1}^{\infty} A \cap \left(B(0, n+1) - B(0, n)\right)$.

Pf of ②: Case 1: A bdd $\longrightarrow$ Use part 1.

Case 2: Write $A = \bigcup_{1}^{\infty} \underbrace{A \cap \left(B(0, n+1) - B(0,n) \right)}_{A_n}$.

Case 1 $\Rightarrow \exists\, U_n \supseteq A_n$ & $K_n \subseteq A_n \ni U_n$ open, K_n cpt & $\mu(U_n - K_n) < \frac{\varepsilon}{2^n}$

Let $U = \bigcup_{1}^{\infty} U_n \longleftarrow$ open.

& $C = \bigcup_{1}^{\infty} K_n \longleftarrow$ closed

$\Big\} \longrightarrow$ done!

5.3. **Non-measurable sets.**

Theorem 5.13. *There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{L}(R)$.*

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)
(2) Let $E \subset \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.
(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.
(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.
(5) $\lambda(E) > 0$