5. Abstract measures

5.1. Dynkin systems.

Question 5.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

▷ Clearly need Π to be closed under intersections.

Thm: $\Pi \to$ closed under int

$\mu, \nu \to 2$ finite measures

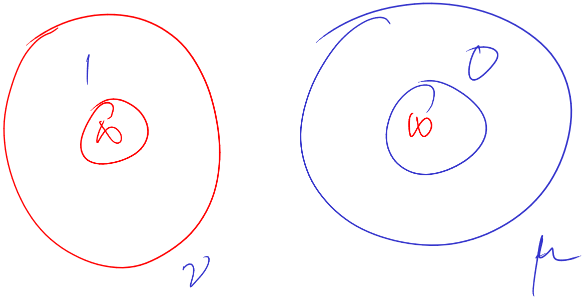
$\mu = \nu$ on Π

$\Rightarrow \mu = \nu$ on $\sigma(\Pi)$

$A \subseteq B$

Must

$$\left.\begin{array}{l}\mu(A) = \nu(A)\\ \mu(B) = \nu(B)\end{array}\right\} \to \mu(B-A) = \nu(B-A)\ ?$$



Question 5.2. *Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ-algebra?*

$\boxed{(\mu, \nu \text{ finite})}$

- ▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$? NO $\longrightarrow$ Yes if ① $A \cap B = \phi$ ② $A \subseteq B$ ~~③ $\mu(A \cap B) = \nu(A \cap B)$~~
- ▷ If $A \subseteq B$, $A, B \in \Lambda$, must $\boxed{B - A \in \Lambda?}$

  Yes. $\mu(B - A) = \mu(B) - \mu(A)$
  $= \nu(B) - \nu(A)$
- ▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

  $\Rightarrow \mu\left(\bigcup_1^\infty A_i\right) = \lim_{i \to \infty} \mu(A_i) = \lim_{i \to \infty} \nu(A_i) = \nu\left(\bigcup_1^\infty A_i\right)$

  Yes

Definition 5.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if:

(1) $X \in \Lambda$

(2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$.

(3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^\infty A_n \in \Lambda$.

Definition 5.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π-system if whenever $A, B \in \Pi$, we have $A \cap B \in \Pi$.

Lemma 5.5 (Dynkin system lemma). *If Π is a π-system, and $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \sigma(\Pi)$.*

Corollary 5.6. *If μ, ν are finite measures such that $\mu = \nu$ on Π, and Π is closed under intersections, then $\mu = \nu$ on $\sigma(\Pi)$.*

(and assume $X \in \Pi$).

Pf: $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$.

Λ is a λ-system (prev slide)

$\Lambda \supseteq \Pi$ (assumption).

Dynkin $\Rightarrow$ $\Lambda \supseteq \sigma(\Pi)$

QED.

Proof of Lemma 5.5

(1) ~~The arbitrary intersection~~ of λ-systems is a λ-system. So it make sense to talk about $\lambda(\Pi)$.

$\lambda(\Pi) =$ λ-system gen by Π

(2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.

(3) If Λ is *both* a π-system and a λ-system, then Λ is a σ-algebra.

Pf of (3): NTS $\to$ If $A, B \in \Lambda$ then $A \cup B \in \Lambda$.

Pf: $A \cap B \in \Lambda$ (Λ is a π-sys)

$A - (A \cap B) \in \Lambda$
$B - (A \cap B) \in \Lambda$ $\to$ (Λ is a λ-sys)

$(X - (A - B)) - B$ ✓ Complex

$A, B \in \Lambda \Rightarrow A^c, B^c \in \Lambda$

$\Rightarrow A^c \cap B^c \in \Lambda$ (π-sys)

$\Rightarrow A \cup B = (A^c \cap B^c)^c \in \Lambda$ QED.

Since Λ is closed under inc countable unions

$\Rightarrow \Lambda$ is a σ-alg.

(4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.

(5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. Then Λ_C is a λ-system.

Pf: ① $X \in \Lambda_C$ (Yes: $X \cap C \in \lambda(\Pi)$? ← Yes.)

② $A, B \in \Lambda_C$, $A \subseteq B$. NTS $B - A \in \Lambda_C$.

i.e NTS $(B-A) \cap C \in \lambda(\Pi)$

$$(B-A)\cap C = \underbrace{\underbrace{(B\cap C)}_{\in \lambda(\Pi)} - \underbrace{(A \cap C)}_{\in \lambda(\Pi)}}_{\in \lambda(\Pi)}$$

③ Inc unions → True (check).

(6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.

1. ▷ Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.

▷ For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

Pf: $\Lambda_D \supseteq \Pi$ (Π is a π-system)

Know Λ_D is a λ-sys (step 5). $\Rightarrow \Lambda_D \supseteq \lambda(\Pi)$. QED.

$\Rightarrow \forall B \in \lambda(\Pi)$, $B \in \Lambda_D \Rightarrow B \cap D \in \lambda(\Pi)$.

Pf: Λ_B is a λ-system (step 5)

$\Lambda_B \supseteq \Pi$ ✓

$\forall D \in \Pi$ must $D \cap B \in \lambda(\Pi)$?

Yes by

$\Rightarrow \underbrace{\Lambda_B \supseteq \lambda(\Pi)}$.

$\Rightarrow \forall C \in \lambda(\Pi)$, $C \in \Lambda_B \Rightarrow B \cap C \in \lambda(\Pi)$

QED.

5.2. Regularity of measures.

μ is a measure $(X, \mathcal{B}(X))$

Definition 5.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(Radon) (1) For all compact sets K, we have $\mu(K) < \infty$.

(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$. (inner regular)

(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$. (outer regular).

Motivation:

- ▷ Approximation of measurable functions by continuous functions
- ▷ Differentiation of measures
- ▷ Uniqueness in the Riesz representation theorem

Question 5.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?* (False in general.)

$\inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U)$

$\sup_{K \subseteq U} \mu(K)$ $(K \not\subseteq A)$

True when $X = \mathbb{R}^d$ (closed sets)

Thm: X cpt & μ finite then μ is regular

Remark 5.9. (1) If $X = \mathbb{R}^d$, and μ is regular, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\mu(U - C) < \varepsilon$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. $\square$

Theorem 5.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. Then μ is regular. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ closed such that $\mu(U - K) < \varepsilon$.*