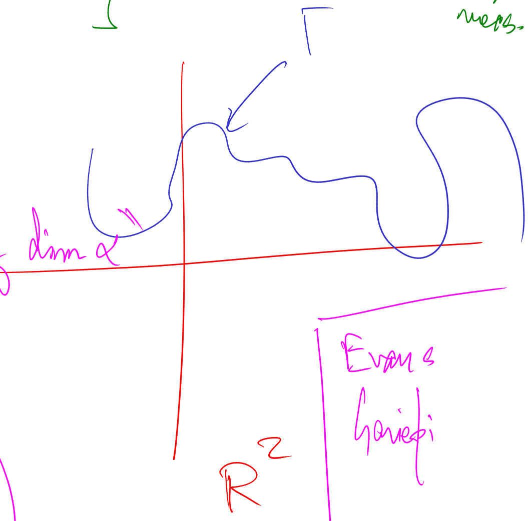
$\lambda^*(A) = \inf \left\{ \sum_i \pi r_i^{2} \;\middle|\; \bigcup_{i=1}^{\infty} B(x_i, r_i) \supseteq A \right\}$ ← should be = Lebesgue meas.

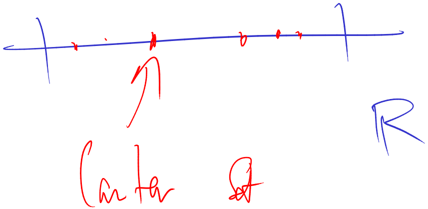
($\sum_i \pi r_i^2$ crossed out and replaced by $\sum_i r_i^\alpha$)

$\lambda^*$ = Lebesgue meas on $\mathbb{R}^2$. ($\alpha > 0$) → "Hausdorff measure of dim $\alpha$"

Q: $\lambda^*(\Gamma) = 0$




$0 = \lambda^*(C)$ →




Evans Gariepy

**Definition 4.25.** Define the *Lebesgue $\sigma$-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \cap \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

**Definition 4.26.** Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

*Remark* 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a $\sigma$-algebra, and $\lambda$ is a measure on $\mathcal{L}$.

**Question 4.28.** *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

Yes

**Proposition 4.29.** *If $I \subseteq \mathbb{R}^d$ is a cell, then $I \in \mathcal{L}(\mathbb{R}^d)$.* (i.e. $\forall A$, $\lambda^*(A) = \lambda^*(A \cap I) + \lambda^*(A \cap I^c)$)

*Proof:* Only WTS $\lambda^*(A) \geq \lambda^*(A \cap I) + \lambda^*(A \cap I^c)$

Pick $\varepsilon > 0$. Pick a cell $J_\varepsilon \subseteq I$ + $d(J_\varepsilon, I^c) > 0$ and $\lambda^*(I - J_\varepsilon) < \varepsilon$

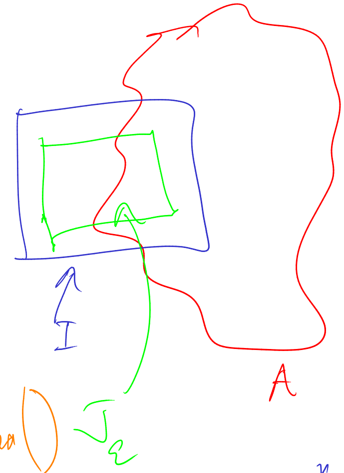
$$\Rightarrow \lambda^*(A) \geq \lambda^*(A \cap (J_\varepsilon \cup I^c)) \overset{\text{(sep add)}}{=} \lambda^*(A \cap J_\varepsilon) + \lambda^*(A \cap I^c)$$

~~Hope~~ $\geq \lambda^*(A \cap I) - \varepsilon$

$$\lambda^*(A \cap I) \leq \lambda^*(A \cap J_\varepsilon) + \lambda^*(A \cap (I - J_\varepsilon)) \quad \text{(sub add)}$$

$$\leq \quad '' \quad + \lambda^*(I - J_\varepsilon)$$

$$\leq \quad '' \quad + \varepsilon \Rightarrow$$




$$\therefore \lambda^*(A) \geq \lambda^*(A \cap I) - \varepsilon + \lambda^*(A \cap I^c)$$

QED

**Proposition 4.30.** $\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d)$.

*Remark* 4.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Null sets

Pf: Any open set is a ctable union of cells

$\Rightarrow \mathcal{L}(\mathbb{R}^d) \supseteq$ all open sets

$\Rightarrow \mathcal{L}(\mathbb{R}^d) \supseteq \sigma(\quad) = \mathcal{B}(\mathbb{R}^d)$. QED.

$N \in \mathcal{N}$. $(\lambda^*(N) = 0)$ Q: $N \in \mathcal{L}(\mathbb{R}^d)$? ✓

NTS

$\lambda^*(A) \geq \underbrace{\lambda^*(A \cap N)}_{0} + \underbrace{\lambda^*(A \cap N^c)}_{\leq \lambda^*(A)}$ ← True

Here are two results that will be proved later:

**Theorem 4.32.** $\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d)$. *(In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.)*

**Theorem 4.33.** $\mathcal{L}(\mathbb{R}^d) \subsetneq \mathcal{P}(\mathbb{R}^d)$.

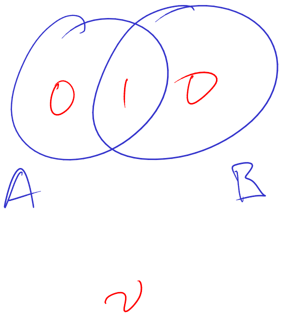
(Proofs later)

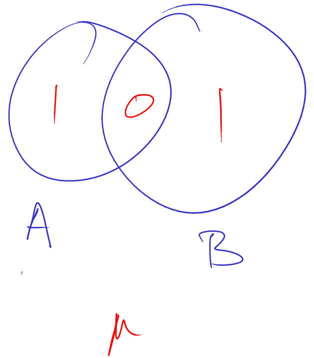
**Theorem 4.34** (Uniqueness)**.** *If $\mu$ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.*

**Question 4.35.** *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose $\mu, \nu$ are two measures which agree on $\mathcal{E}$. Must they agree on $\sigma(\mathcal{E})$?* $\sigma(\mathcal{E})$

$\mathcal{E} = \{A, B\}$

NO!



Pf of 4.34: Know $\mu(I) = \lambda(I)$ $\forall$ cells $I$.

Claim: $\mu \leq \lambda$

$A \in \mathcal{B}(\mathbb{R}^d)$. $A \subset \bigcup_1^\infty I_k \Rightarrow \mu(A) \leq \sum_1^\infty \mu(I_k) = \sum_1^\infty \lambda(I_k)$

$\Rightarrow \mu(A) \leq \inf\{\sum_1^\infty \ell(I_k) \mid \bigcup_1^\infty I_k \supseteq A\} = \lambda(A) \Rightarrow$ Claim.

Claim 2: Say $E$ is bdd. Then $\lambda(E) \leq \mu(E)$

Pf: Find a cell $I$ s.t. $I \supseteq E$. $\mu(I) - \mu(E) = \mu(I - E) \leq \lambda(I - E) = \lambda(I) - \lambda(E)$

$\Rightarrow \mu(E) \geq \lambda(E)$ QED.

Claim 3: $\forall E$, $\lambda(E) \leq \mu(E)$

Pf: Write $E = \bigcup_1^\infty E_i$, $E_i$ are disj & bdd & use claim 2.

QED.

# 5. Abstract measures

## 5.1. Dynkin systems.

**Question 5.1.** *Say $\mu, \nu$ are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

▷ Clearly need $\Pi$ to be closed under intersections.

**Question 5.2.** *Let* $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. *Must* $\Lambda$ *be a* $\sigma$*-algebra?*

- ▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$? NO
- ▷ If $A \subseteq B$, $A, B \in \Lambda$, must $B - A \in \Lambda$? ($\mu$, $\nu$, finite, Yes)
- ▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

$$\mu(B-A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B-A)$$