Proposition 4.18 (Translation invariance). *For all $A \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $\lambda^*(A) = \lambda^*(\alpha + A)$.*

Last time: $\lambda^* \to$ Lebesgue OUTER measure. $\to \lambda^*(A) = \inf\left\{ \sum_k \ell(I_k) \,\middle|\, \bigcup I_k \supseteq A \text{ \& } I_k\text{'s a cells} \right\}$

$\hookrightarrow \lambda^*\left(\bigcup_{i}^{\infty} A_i\right) \le \sum_{i}^{\infty} \lambda^*(A_i)$ (sub add)

$$\boxed{\text{Today: Want a meas out of } \lambda^*}$$

$\ell(I+\alpha) = \ell(I)$

"Haar Measure"

4.2. Carathéodory Extension. Our goal is to start with an *outer measure*, and restrict it to a *measure*.

Definition 4.19. We say μ^* is an outer measure on X if:

(1) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, and $\boxed{\mu^*(\emptyset) = 0}$.
(2) If $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$.
(3) If $A_i \subseteq X$ (not necessarily disjoint), then $\mu^*(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \mu^*(A_i)$. (countable sub-additivity)

Example 4.20. Any measure is an outer measure.

Example 4.21. The Lebesgue outer measure is an outer measure.

Theorem 4.22 (Carathéodory extension). *Let* $\Sigma \overset{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\ \forall A \subseteq X\}$. *Then* Σ *is a* σ*-algebra, and* μ^* *is a measure on* (X, Σ).

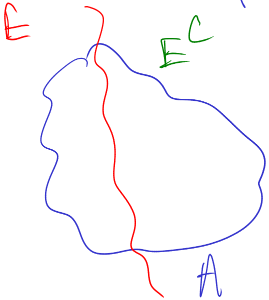
Remark 4.23. Clearly $\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \geqslant \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c. If E is nice, hopefully the overlap is small.

① $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ $\forall A, E$ (sub add).

② For nice sets E ($E \in \Sigma$) also have $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

↪ λ^*; A ← cover by cells: $A \subseteq \bigcup_{k=1}^{\infty} I_k$: $\mu^*(A) \geq \sum \ell(I_k) - \varepsilon$.



$\{I_k\} = \{J_k\} \cup \{J'_k\} \cup \{J''_k\}$

$J_k \cap E \neq \emptyset$ $\quad J'_k \cap E^c \neq \emptyset$ $\quad J''_k \cap E \neq \emptyset$

$\cancel{J_k \cap E^c = \emptyset}$ $\quad \cancel{J'_k \cap E = \emptyset}$ $\quad J''_k \cap E^c \neq \emptyset$

$$\mu^*(A) \geq \sum \ell(I_k) - \varepsilon = \sum \ell(J_k) + \sum \ell(J_k') - \sum \ell(J_k'') - \varepsilon$$

$$\geq \underline{\mu^*(A \cap E)} + \underline{\mu^*(A \cap E^c)} - \sum \ell(J_k') - \varepsilon$$

Hope this is small!

Proof of Theorem 4.22 (Carathéodory)

(1) $\emptyset \in \Sigma$. → NTS $\mu^*(A) = \underbrace{\mu^*(A \cap \emptyset)}_{0} + \underbrace{\mu^*(A \cap X)}_{\mu^*(A)}$

(2) $E \in \Sigma \implies E^c \in \Sigma$.

(3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \ldots, E_n \in \Sigma \implies \cup_1^n E_i \in \Sigma$.)

$E \in \Sigma \Rightarrow \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$

$E^c \in \Sigma$? $\mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c) = \mu^*(A)$ ✓

③ NTS $E \cup F \in \Sigma$.

NTS $\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c) = \mu^*(A) \quad \forall A$

LHS

$= \mu^*(A \cap (E \cup F) \cap E) + \mu^*(A \cap (E \cup F) \cap E^c) + \mu^*(A \cap (E \cup F)^c) \quad (\because E \in \Sigma)$.

$= \mu^*(A \cap E) + \underbrace{\mu^*(A \cap F \cap E^c) + \mu^*(A \cap E^c \cap F^c)}_{\mu^*(A \cap E^c) \quad (\because F \in \Sigma)}$

$= \mu^*(A)$ QED

(4) If $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\boxed{\mu^*(A \cap (\cup_1^n E_i)) = \sum_1^n \mu^*(A \cap E_i).}$

NTS $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F)$ $\left(\forall A \subseteq X, \; E, F \subset \Sigma \atop E \cap F = \phi\right)$

Pf: $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap (E \cup F) \cap E) + \mu^*(\underline{\hspace{3cm}} \cap E^c)$

$= \mu^*(A \cap E) + \mu^*(A \cap F)$

QED.

(5) $\boxed{\Sigma \text{ is closed under countable } \textit{disjoint} \text{ unions,}}$ and $\boxed{\mu^* \text{ is countably additive on } \Sigma.}$ $(\Longrightarrow \Sigma$ is a σ-alg$)$

$\Rightarrow \mu^*|_\Sigma$ is a meas!

Proof: Let $E_1, E_2, \ldots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

NTS $\bigcup_1^\infty E_i \in \Sigma$. $\Leftrightarrow$ NTS $\forall A$, $\mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c) = \mu^*(A)$.

$$\mu^*(A) = \mu^*(A \cap (\bigcup_1^N E_i)) + \mu^*(A \cap (\bigcup_1^N E_i)^c) \qquad (\because \bigcup_1^N E_i \in \Sigma)$$

$$\geq \sum_1^N \mu^*(A \cap E_i) + \mu^*(A \cap (\bigcup_1^{\cancel{N}\,\infty} E_i)^c) \qquad \forall N \qquad \text{send } N \to \infty$$

$$\Rightarrow \mu^*(A) \geq \sum_1^\infty \mu^*(A \cap E_i) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c)$$

$$\geq \mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c) \geq \mu^*(A)$$

$$\Rightarrow \mu^*(A) = \mu^*(A \cap (\bigcup_1^\infty E_i)) + \mu^*(A \cap (\bigcup_1^\infty E_i)^c)$$

$\Rightarrow \bigcup_1^\infty E_i \in \Sigma$. QED.

$\&\ \mu^*(A \cap (\bigcup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$

Remark 4.24. Note, the above shows $\boxed{\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i).}$

Definition 4.25. Define the *Lebesgue σ-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) \cap \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 4.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

Remark 4.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$.

Question 4.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*