Definition 3.14. Let Σ be a σ-algebra on X. We say μ is a (positive) measure on (X, Σ) if:

(1) $\mu\colon \Sigma \to [0, \infty]$
(2) $\mu(\emptyset) = 0$
(3) (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

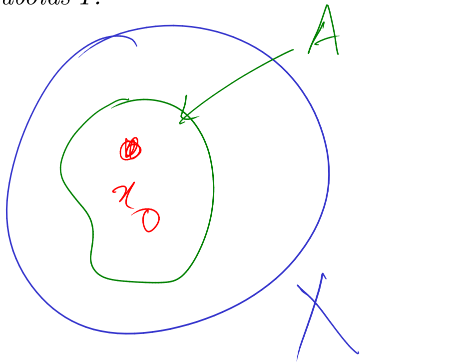
Question 3.15. *Is the second assumption necessary?* YES

Question 3.16. *Let $\mu(A)$ = cardinality of A. Is μ a measure?*

$\mu(A) = \begin{cases} \#\text{ elms in } A & (\text{if } A \text{ is fin}) \\ +\infty \end{cases}$

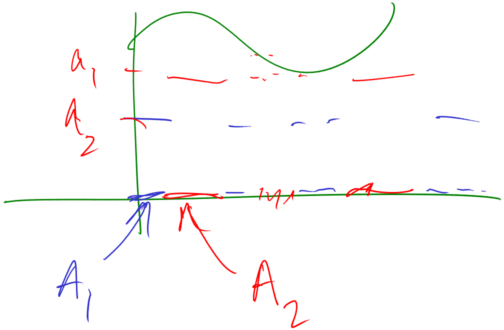
Question 3.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?* ← measure (YES).

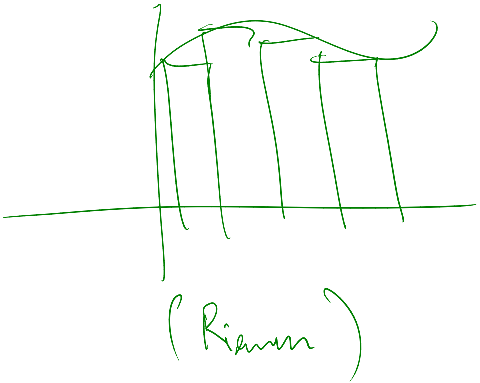
Theorem 3.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \mathrm{vol}(I)$ for all cuboids I.*



- **Goal:** Define $\int_X f\,d\mu$ (the Lebesgue integral).
- **Idea:**
  - ▷ Say $s : X \to \mathbb{R}$ is such that $s = \sum_1^N a_i \mathbf{1}_{A_i}$, for some $a_i \in \mathbb{R}$, $A_i \in \Sigma$. (Called *simple functions*.)
  - ▷ Define $\int_X s\,d\mu = \sum_1^N a_i \mu(A_i)$.
  - ▷ If $f \geqslant 0$, define $\int_X f\,d\mu = \sup_{s \leqslant f} \int_X s\,d\mu$.
- Will do this after constructing the Lebesgue measure.

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$





4. Construction of the Lebesgue Measure

4.1. Lebesgue Outer Measure.

[a, b] (a, b), (a, b]. [a, b)

Definition 4.1. We say $I \subseteq \mathbb{R}$ is a *cell* if I is a finite interval. Define $\ell(I) = \sup I - \inf I$.

Definition 4.2. We say $I \subseteq \mathbb{R}^d$ is a *cell* if it is a product of cells. If $I = I_1 \times \cdots \times I_d$, then define $\ell(I) = \prod_{i=1}^d \ell(I_i)$.

Remark 4.3. $\ell(I) = \ell(\mathring{I}) = \ell(\bar{I})$.

Remark 4.4. $\emptyset = \prod_1^d (a, a)$, and so $\ell(\emptyset) = 0$.

Remark 4.5. For all $\alpha \in \mathbb{R}^d$, $\ell(I) = \ell(I + \alpha)$.

Theorem 4.6. *There exists a (unique) measure λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\lambda(I) = \ell(I)$ for all cells I.*

Question 4.7. *How do you extend ℓ to other sets?*

Definition 4.8 (Lebesgue outer measure)**.** Given $A \subseteq \mathbb{R}^d$, define $\lambda^*(A) = \inf\left\{ \sum_1^\infty \ell(I_k) \;\middle|\; A \subseteq \bigcup_1^\infty I_k, \text{where } I_k \text{ is a cell} \right\}$.

Remark 4.9. Some authors use m^* instead of λ^*.

Remark 4.10. λ^* is defined on $\mathcal{P}(\mathbb{R}^d)$; but only "well behaved" on a σ-algebra.

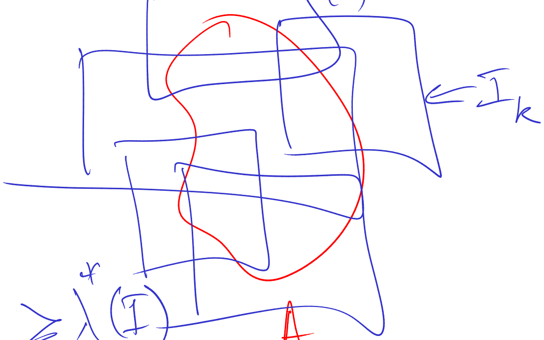
Question 4.11. *What is $\lambda^*(\emptyset)$? What is $\lambda^*(\mathbb{R}^d)$?*

0 $\qquad +\infty.$

$\lambda(A) \le \sum \ell(I_k)$

① If $A \subseteq B$, $\lambda^*(A) \le \lambda^*(B)$

② $\lambda^*(I) = \ell(I)$ (IOU)



$\lambda^*(\mathbb{R}^d) \ge \lambda^*(I)$

Proposition 4.12. *If $E \subseteq F$, then $\lambda^*(E) \leqslant \lambda^*(F)$.*

Proposition 4.13. *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then $\lambda^*(\cup_1^\infty E_i) \leqslant \sum_1^\infty \lambda^*(E_i)$.*

Pf: $\forall i$, $\forall \varepsilon > 0$ $\exists$ cells $I_{i,k}$ s.t. $\lambda^*(E_i) \geq \sum_k \ell(I_{i,k}) - \frac{\varepsilon}{2^i}$

$$\left(E_i \subseteq \bigcup_k I_{i,k}\right).$$

Clearly $\bigcup_{i,k} I_{i,k} \supseteq \bigcup_{i=1}^\infty E_i$

$$\Rightarrow \lambda^*\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i,k} \ell(I_{i,k}) \leq \sum_{i=1}^\infty \left(\lambda^*(E_i) + \frac{\varepsilon}{2^i}\right) = \sum \lambda^*(E_i) + \varepsilon$$

QED.

Proposition 4.14. *Let $A, B \subseteq \mathbb{R}^d$, and suppose $\boxed{d(A,B) > 0}$. Then $\lambda^*(A \cup B) = \lambda*(A) + \lambda*(B)$.*

Proof: Only need to show $\lambda^*(A \cup B) \geqslant \lambda^*(A) + \lambda^*(B)$. If $\lambda^*(A \cup B) = \infty$, we are done, so assume $\lambda^*(A \cup B) < \infty$.

Pick $\varepsilon > 0$.

$\exists$ cells I_k $\ni$ $A \cup B \subseteq \bigcup_1^\infty I_k$ & $\lambda^*(A \cup B) \geqslant \sum \ell(I_k) - \varepsilon$

Let $\{I_k\} = \{J_k\} \cup \{J_k'\}$ $\ni$ $J_k \cap A \neq \phi$ & $J_k \cap B = \phi$ $J_k' \cap B \neq \phi$ & $J_k' \cap A = \phi$

(Subdivide I_k if necessary $\ni$ diam $(I_k) < d(A,B)$)

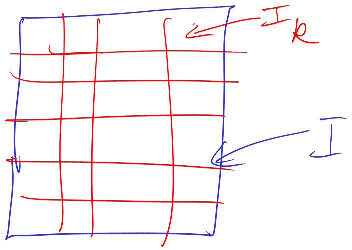
$\left.\begin{array}{l} \lambda^*(A) \leq \sum \ell(J_k) \\ \lambda^*(B) \leq \sum \ell(J_k') \end{array}\right\}$ $\lambda^*(A) + \lambda^*(B) = \sum \ell(J_k) + \sum \ell(J_k') = \sum \ell(I_k) \leq \underline{\lambda^*(A \cup B)} + \varepsilon$

QED.

Proposition 4.15. *If $I \subseteq \mathbb{R}^d$ is a cell, then $\lambda^*(I) = \ell(I)$.*

Lemma 4.16. *If $\{I_k\}$ divide I by hyperplanes, then $\sum \ell(I_k) = \ell(I)$.* (Dist law).

Lemma 4.17. $\lambda^*(A) = \inf\{\sum \ell(I_i) \mid A \subseteq \cup I_k,$ *and I_k are all* open *cells*$\}$.



Pf: $\varepsilon > 0$. $\exists \{I_j\} \ni A \subseteq \bigcup I_j$

& $\lambda^*(A) \geq \sum \ell(I_j) - \varepsilon$

Let $J_k \supseteq I_k$ & $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$ & J_k open.

$\Rightarrow \quad \sum \ell(J_k) \leq \sum \ell(I_k) + \varepsilon \leq \lambda^*(A) + 2\varepsilon$

QED.

Proof of Proposition 4.15: Suppose first I is closed (hence compact). Pick $\varepsilon > 0$.

NTS $\ell(I) = \lambda^*(I)$. Know $\lambda^*(I) \leq \ell(I)$. NTS $\lambda^*(I) \geq \ell(I)$.

Pick $\varepsilon > 0$. $\exists$ open cells I_k st $I \subseteq \bigcup_1^\infty I_k$ & $\lambda^*(I) \geq \sum_1^\infty \ell(I_k) - \varepsilon$

Use finite subcover. $\lambda^*(I) \geq \sum_1^N \ell(I_k) - \varepsilon \geq \ell(I) - \varepsilon$. QED.

extend faces of each cell to divide I by hyperplanes.