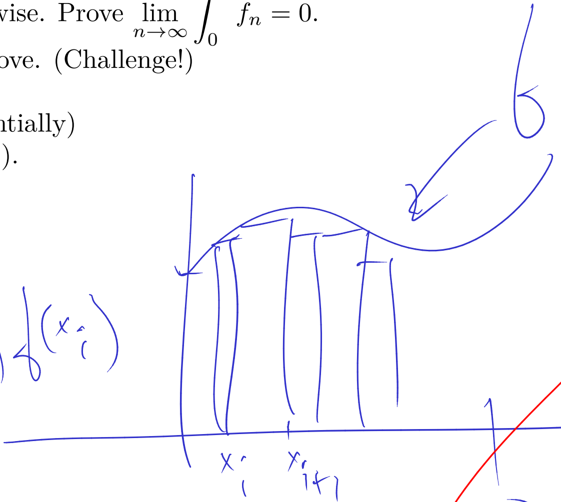
1. **Syllabus Overview**

- Class website and full syllabus: `http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/720-measure`
- TA: Lantian Xu <lxu2@andrew.cmu.edu>
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterm: Fri Oct 9th (90 mins, self proctored, can be taken any time)
- **Zoom lectures:**
  - ▷ Please enable video. (It helps me pace lectures).
  - ▷ Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - ▷ If I get disconnected, check your email for instructions.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Bottom 20% homework is dropped from your grade (personal emergencies, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - ▷ Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - ▷ Self proctored: Zoom call (invite me). Record yourself, and your screen to the cloud.
  - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.

- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
    - Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 40% homework, 20% midterm, 40% final.

2. Sigma Algebras and Measures

- **Motivation:** Suppose $f_n \colon [0,1] \to [0,1]$, and $(f_n) \to 0$ pointwise. Prove $\lim_{n\to\infty} \int_0^1 f_n = 0$.
  - ▷ Simple to state using Riemann integrals. Not so easy to prove. (Challenge!)
  - ▷ Will prove this using Lebesgue integration.
    - – Riemann integration: partition the domain (count sequentially)
    - – Lebesgue integration: partition the range (stack and sort).
- **Goal:**
  - ▷ Develop Lebesgue integration.
  - ▷ Need a notion of "measure" (generalization of volume)
  - ▷ Need "σ-algebras".



Area

Volume

"measure"

Riemann: $\int_0 f \approx \sum (x_{i+1} - x_i) f(x_i)$

Lebesgue: Say $f = \begin{cases} a_1 & \text{on } A_1 \\ a_2 & \text{on } A_2 \end{cases}$ $A_1, A_2 \dots$ disj.

$\leadsto \int_0 f \approx \sum a_i$ "length" (A_i)

- Why σ-algebras?

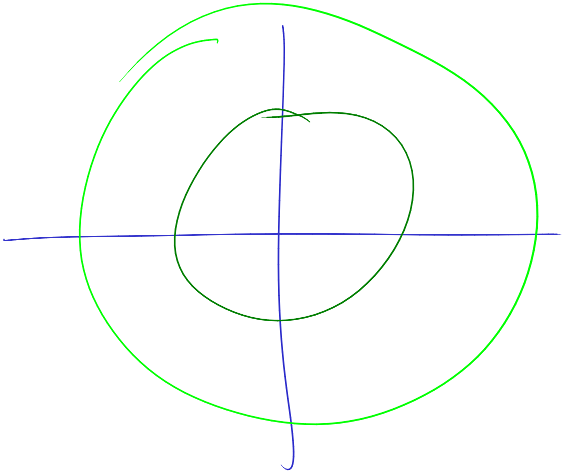
$n < 5$

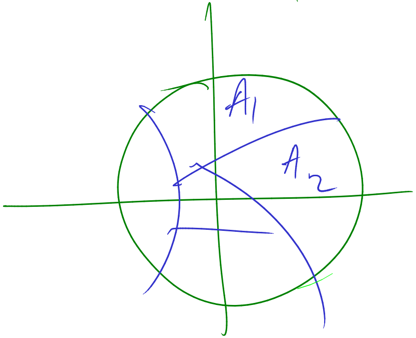
Theorem 2.1 ((Banach Tarski)). *There exists $n \in \mathbb{N}$, sets $A_1, \ldots, A_n \subseteq B(0,1) \subseteq \mathbb{R}^3$ such that:*

$\mathbb{R}^3$

(1) $A_1, \ldots, A_n$ partition $B(0,1)$.
(2) There exist isometries R_i such that $R_1(A_1), \ldots, R_n(A_n)$ partition $B(0,2)$.

- How do you explain this?





Definition 2.2 (σ-algebra)**.** Let X be a set. We say $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra on X if:

(1) Nonempty: $\emptyset \in \Sigma$
(2) Closed under compliments: $A \in \Sigma \implies A^c \in \Sigma$.
(3) Closed under countable unions: $A_i \in \Sigma \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

$A_1, A_2, \dots \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$

Remark 2.3. Any σ-algebra is also closed under countable intersections.

Demorgan's law.

Question 2.4. *Is $\mathcal{P}(X)$ is a σ-algebra?*

Question 2.5. *Is $\Sigma \stackrel{\text{def}}{=} \{\emptyset, X\}$ is a σ-algebra?*

Question 2.6. *Is $\Sigma = \{A \mid |A| < \infty$ or $|A^c| < \infty\}$ a σ-algebra?*

NO

X is inf.

Question 2.7. *Is $\Sigma = \{A \mid$ either A or A^c is finite or countable$\}$ a σ-algebra?*

Yes.

(finite is countable)

Proposition 2.8. *If $\forall \alpha \in \mathcal{A}$, Σ_α is a σ-algebra, then so is $\bigcap_{\alpha \in \mathcal{A}} \Sigma_\alpha$.* Yes.

Definition 2.9. If $\mathcal{E} \subseteq \mathcal{P}(X)$, define $\boxed{\sigma(\mathcal{E})}$ to be the intersection of all σ-algebras containing $\mathcal{E}$.

Remark 2.10. $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.

Definition 2.11. Suppose X is a topological space. The *Borel σ-algebra on X* is defined to be the σ-algebra generated by all open subsets of X. Notation: $\mathcal{B}(X)$.

Question 2.12. *Can you get $\mathcal{B}(X)$ by taking all countable unions / intersections of open and closed sets?*

Question 2.13. *Is $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$?*

$X = \mathbb{R}$. $\mathcal{E} = \{\text{open sets}\}$ $\quad \mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$

Guess: $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R}) = \{\text{open sets}\} \cup \{\text{closed sets}\} \cup G_\delta \cup F_\sigma \cup G_{\delta\sigma} \cup F_{\sigma\delta} \cup \ldots$

(Not enough!)

Definition 2.14. Let Σ be a σ-algebra on X. We say μ is a (~~positive~~) measure on (X, Σ) if:

(1) $\mu\colon \Sigma \to [0, \infty]$

(2) $\mu(\emptyset) = 0$

(3) (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 2.15. *Is the second assumption necessary?*

Question 2.16. *Let $\mu(A) =$ cardinality of A. Is μ a measure?*

Question 2.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?*

Theorem 2.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \operatorname{vol}(I)$ for all cuboids I.*

$A \in \Sigma \Rightarrow A \subseteq X, \mu(A) \in [0, \infty]$

$E \in \Sigma$

$\cancel{\mu(E)} = \mu(E \cup \emptyset) = \cancel{\mu(E)} + \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0$