21-720 Measure Theory Final Part I.

2020-12-14

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not, *however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 90 mins. All questions are worth 10 points. Good luck ⌣̈.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure of a Lebesgue measurable set $A \subseteq \mathbb{R}^d$ will be denoted by $|A|$, and integrals with respect to the Lebesgue measure will be denoted as $\int_{\mathbb{R}^d} f \, d\lambda$ or $\int_{\mathbb{R}^d} f(x) \, dx$.

1. Suppose $\mu(X) > 0$, and $f \colon X \to (0, \infty)$ is measurable. Must $\int_X f \, d\mu > 0$? Prove it, or find a counter example.

2. Let μ, ν be two finite signed Borel measures on $\mathbb{R}^d$, and π be the product measure of μ and ν on $\mathbb{R}^d \times \mathbb{R}^d$. Let $\varphi \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be defined by $\varphi(x, y) = x + y$, and let $\eta = \varphi^*(\pi)$ be the push-forward of π (i.e. for every $A \in \mathcal{B}(\mathbb{R}^d)$, $\eta(A) = \pi(\varphi^{-1}(A))$). Estimate the total variation of η in terms of the total variations of μ and ν.

3. Suppose $\alpha \in (0, 1)$ and μ is a σ-finite positive Borel measure on $\mathbb{R}^d$ such that $\mu(B(x, r)) \leqslant |B(x, r)|^\alpha$ for every $x \in \mathbb{R}^d$, $r > 0$. Must μ be absolutely continuous with respect to the Lebesgue measure? Prove it, or find a counter example.

4. True or false: There exists $q \in [1, \infty]$ a bounded linear operator $\mathcal{F} \colon L^{4/3}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ such that $\mathcal{F} f = \hat{f}$ for all $f \in C_c^\infty(\mathbb{R}^d)$. Prove or disprove it. [Hint: Play with $(f * f)^\wedge$ for $f \in C_c^\infty(\mathbb{R}^d)$.]

21-720 Measure Theory Final Part II.

2020-12-14

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not, *however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 90 mins. All questions are worth 10 points. Good luck ⌣̈.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure of a Lebesgue measurable set $A \subseteq \mathbb{R}^d$ will be denoted by $|A|$, and integrals with respect to the Lebesgue measure will be denoted as $\int_{\mathbb{R}^d} f \, d\lambda$ or $\int_{\mathbb{R}^d} f(x) \, dx$.

1. Suppose $f, f_n \colon X \to \mathbb{R}$ are measurable functions such that $(f_n) \to f$ almost everywhere. Suppose further for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ and $F_\varepsilon \in \Sigma$ such that $\mu(F_\varepsilon) < \infty$ and for all $n \geqslant n_\varepsilon$ we have $|f_n - f| \leqslant \varepsilon$ on F_ε^c. Must $(f_n) \to f$ in measure? Prove it, or find a counter example.

2. Let $d \geqslant 2$. Given $x \in \mathbb{R}$ define $R_x = (-\infty, x] \times \mathbb{R}^{d-1} \subseteq \mathbb{R}^d$. True or false:

   $$\text{If } f \in L^1(\mathbb{R}^d), \text{ then the function } x \mapsto \int_{R_x} f \, d\lambda \text{ is continuous everywhere.}$$

   Prove it, or find a counter example.

3. Let $d \geqslant 2$. Given $x \in \mathbb{R}$ define $R_x = (-\infty, x] \times \mathbb{R}^{d-1} \subseteq \mathbb{R}^d$. True or false:

   $$\text{If } f \in L^1(\mathbb{R}^d), \text{ then the function } x \mapsto \int_{R_x} f \, d\lambda \text{ is differentiable almost everywhere.}$$

   Prove it, or find a counter example.

4. Let μ be a positive finite measure on $\mathbb{R}^d$ which is mutually singular to the Lebesgue measure. Must

   $$\lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = \infty \,, \quad \text{for } \mu\text{-almost every } x \in \mathbb{R}^d \,.$$

   Prove it, or find a counter example.