## Measure Theory: Final.

Dec 12, 2014

- This is a closed book test. No calculators or computational aids are allowed.
- You have 3 hours. The exam has a total of 8 questions and 70 points.
- You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **CLEARLY** state the result you are using.

Unless otherwise stated, we always assume the underlying measure space is  $(X, \Sigma, \mu)$  and  $\mu$  is a positive measure. The Lebesgue measure on  $\mathbb{R}^d$  will be denoted by  $\lambda$ .

- 10 1. Given  $f: X \to \mathbb{R}$  be measurable, and define  $F: \mathbb{R} \to [-\infty, \infty]$  by  $F(x) = \mu(f < x)$ .
  - (a) True or false: If f is measurable, then F is left continuous. Prove it, or find a counter example.
  - (b) True or false: If f is measurable, then F is right continuous. Prove it, or find a counter example.
- 10 2. Let  $\mu$  be a finite measure on X, and  $f: X \to \mathbb{R}$  be measurable. Decide whether the limits

$$\lim_{n \to \infty} \int_X e^{-n|f|} d\mu \quad \text{and} \quad \lim_{n \to \infty} \int_X e^{+n|f|} d\mu$$

necessarily exist. If yes, compute them. Prove your answer. [By convention, if a sequence approaches  $\infty$ , we say the limit exists and is  $\infty$ .]

10 3. Let  $E \subseteq \mathbb{R}^d$  be Lebesgue measurable. True or false:

For any (possibly infinite) collection of balls  $\{B(x_{\alpha}, r_{\alpha})\}_{\alpha \in \mathcal{A}}$  such that

$$\bigcup_{\alpha \in \mathcal{A}} B(x_{\alpha}, r_{\alpha}) \supseteq E, \quad \text{and} \quad \sup_{\alpha \in A} r_{\alpha} < \infty,$$

there exists a (possibly infinite)  $\mathcal{A}' \subseteq \mathcal{A}$  such that the sub-collection  $\{B(x_{\alpha'}, r_{\alpha'})\}_{\alpha' \in \mathcal{A}'}$  is pairwise disjoint and

$$\bigcup_{\alpha' \in \mathcal{A}'} B(x_{\alpha'}, 5r_{\alpha'}) \supseteq E$$

Prove it, or find a counter example.

|10| 4. Let X be a compact metric space, C(X) denote the set of continuous real valued functions on X. True or false:

If 
$$\mu$$
 is a finite signed Borel measure on X, then  $\|\mu\| = \sup \left\{ \int_X f \, d\mu \mid f \in C(X) \text{ and } \sup_X |f| \leq 1 \right\}.$ 

Prove it, or find a counter example. [You may not use the Riesz representation theorem for this question.]

- 10 5. If  $f,g \in L^2(\mathbb{R}^d)$  compute  $(fg)^{\wedge}$  in terms of  $\mathcal{F}f$  and  $\mathcal{F}g$ . Prove it. [Recall for  $f \in L^1(\mathbb{R}^d)$ , we defined  $\hat{f}(\xi) = \int f(x)e^{-2\pi i \langle x,\xi \rangle} dx$  to be the Fourier transform of f, and  $\mathcal{F}$  denotes the extension of the Fourier transform to  $L^2$ . Hint: First compute  $(\hat{f} * \hat{g})^{\wedge}$  if f, g are Schwartz functions.]
  - 6. Given  $f : \mathbb{R}^2 \to \mathbb{R}$ , define  $G, H : \mathbb{R} \to \mathbb{R}$  by

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$$G(x) = \sup_{y \in \mathbb{R}} f(x, y) \quad \text{and} \quad H(x) = \begin{cases} \operatorname{ess\,sup} f(x, y) & \text{if the function } y \mapsto f(x, y) \text{ is Lebesgue measurable,} \\ 0 & \text{otherwise.} \end{cases}$$

Recall, ess sup is the essential supremum, defined by  $\operatorname{ess sup}_{y} f(x, y) = \sup\{z \mid \lambda\{t \mid f(x, t) > z\} > 0\}.$ 

- (a) True or false: If f is Lebesgue measurable, then so is G. Prove it, or find a counter example.
- (b) True or false: If f is Borel measurable, then so is G. No proof required! Incorrect answers are worth no credit, blank answers half credit and correct answers full credit.

(c) True or false: If f is Borel measurable, then so is H. Prove it, or find a counter example.

10 7. For any  $t \in [0,1]$  and  $N \in \mathbb{N}$  define  $\Delta_{N,t} : \mathbb{R} \to \mathbb{R}$  by

$$\Delta_{N,t}(x) = \sup \{ t + \frac{k}{N} \mid k \in \mathbb{Z} \text{ and } t + \frac{k}{N} \leqslant x \}.$$

True or false:

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If 
$$f \in L^1(\mathbb{R})$$
 and  $\operatorname{supp}(f) \subseteq [0, 1]$ , then there exists an increasing sequence of integers  $N_k \to \infty$  such that  $\lim_{k\to\infty} \int_{\mathbb{R}} |f(x) - f(\Delta_{N_k,t}(x))| \, dx = 0$  for almost every  $t \in [0, 1]$ .

Prove it, or find a counter example. [HINT: Play with  $\int_0^1 \int_{\mathbb{R}} |f(x) - f(\Delta_{N,t}(x))| dx dt$ .]

If you've completed the remainder of this exam and have time to spare, here is a fun question. This is for your entertainment only, and *will not influence your grade*.

0 8. Let X be a locally compact metric space, and  $\mu$  a regular Borel measure on X. Suppose  $\mu(\{x\}) = 0$  for every  $x \in X$ . If  $F \in \mathcal{B}(x)$  has finite measure, and  $0 < \alpha < \mu(F)$ , show that there exists  $A \in \mathcal{B}(X)$  such that  $A \subset F$  and  $\mu(A) = \alpha$ .