21-720 Measure Theory: Midterm.

December 10^{th}, 2013

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 8 questions and 35 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are ROUGHLY in order of length / difficulty, and not in the order the material was covered. However, depending on your intuition, you might find a few of the later questions easier. Good luck!*

In this exam, we always assume (X, Σ, μ) is a measure space. We use λ to denote the Lebesgue measure on $\mathbb{R}^d$.

[5] 1. Recall the maximal function of a function f is defined by $Mf(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f| \, d\lambda$. True or false:

If $f \in L^1$, then Mf is measurable.

Prove it, or find a counter-example.

[5] 2. Consider the function $f(x,y) = \frac{xy}{x^4 + y^4}$ when $(x,y) \neq 0$ and $f(0,0) = 0$ defined on the rectangle $[-1,1]^2$. Consider each of the following integrals:

(a) $\int_{-1}^{1} \Big(\int_{-1}^{1} f(x,y) \, d\lambda(x) \Big) \, d\lambda(y)$ (b) $\int_{-1}^{1} \Big(\int_{-1}^{1} f(x,y) \, d\lambda(y) \Big) \, d\lambda(x)$ (c) $\int_{[-1,1]^2} f \, d\lambda$.

For each of the integrals above, decide if they exist (in the extended sense) or not. If yes, compute it. If not, prove it.

[5] 3. Let ϕ_n be an approximate identity. True or false:

For every $f \in L^1(\mathbb{R}^d)$ such that $\hat{f} \in L^1(\mathbb{R}^d)$ the sequence $(\phi_n * f)^\wedge$ is convergent in $L^1(\mathbb{R}^d)$.

Prove, or find a counter example. [Recall that (ϕ_n) is an approximate identity if $\phi_n \geqslant 0$, $\int_{\mathbb{R}^d} \phi_n \, d\lambda = 1$, and for every $\varepsilon > 0$ we have $\lim_{n\to\infty} \int_{|x|>\varepsilon} \phi_n(x) \, dx = 0$.]

[5] 4. Let $g \in L^1(X, \mu)$, and $f : X \to \mathbb{R}$ be measurable. Let (f_n) be a sequence of measurable functions such that $(f_n) \to f$ in measure, and for all $n \in \mathbb{N}$ we have $|f_n| \leqslant g$ almost everywhere. True or false:

The sequence (f_n) necessarily has a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere.

Prove it, or find a counter example.

[5] 5. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Suppose further $f' \in L^2([0,1])$. True or false:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon \sqrt{|x - y|}$.

Prove it, or find a counter example.

[5] 6. True or false:

For every $E \in \mathcal{L}(\mathbb{R})$ with $0 < \lambda(E) < \infty$, we must have $\lim_{n\to\infty} \int_E \cos^2(nx) \, dx$ exists.

Prove it, or find a counter example.

[5] 7. Let $\varphi \in C_c(\mathbb{R}^d)$ be a non-negative radially decreasing function with $\int_{\mathbb{R}^d} \varphi = 1$. True or false:

For all $f \in L^1(\mathbb{R}^d)$, we must have $f * \varphi \leqslant Mf$.

Prove it, or find a counter example. [Recall: We say φ is radially decreasing if there exists a decreasing function $h : [0,\infty) \to \mathbb{R}$ such that $\varphi(x) = h(|x|)$.]

If you've completed the remainder of this exam and have time to spare, here is a fun question. This is for your entertainment only, and *will not influence your grade*.

[0] 8. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $f(x+y) = f(x) + f(y)$, then show f is continuous.