21-370 Discrete Time Finance: Midterm 2.

2020-11-05

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not*, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 60 minutes. The exam has a total of 3 questions and 40 points.*
- *Difficulty wise,* $Q1 \lessapprox Q2 \approx Q3(a) \lessapprox Q3(b)$*. Depending on your intuition, some of you may feel differently, nevertheless, I suggest looking over the entire exam before starting. Good luck* $\ddot\smile$*.*

10 1. Consider the N-period binomial model with parameters $u = 2$, $d = 1/2$, $r = 1/2$, $S_0 = 10$ and $N = 4$. An American option has intrinsic value $G = (G_0, \dots, G_N)$, where $G_n = 0$ if $n \leqslant 1$, and $G_n = \mathbf{1}_{\{5 \leqslant S_n \leqslant 20\}}$ for $n \geqslant 2$. What is the arbitrage free price of this option at time 2 given that the first two coins came up heads? Also determine whether you should exercise this option in this situation (at time 2 given that the first two coins are heads), or wait until the third coin is tossed. Explain why.

10 2. Consider the N-period binomial model with $0 < d < 1 + r < u$, and let S_n denote the stock price at time n. A European option pays $G = g(S_N)$ at maturity time N, where g is some (deterministic) function. Let Δ_n be the number of shares held at time n in the portfolio required to replicate this option. True or false:

   If g is increasing, then for every $n \in \{0, \dots, N\}$, we must have $\Delta_n \geqslant 0$.

   If true, prove it. If false, find a counter example.
   HINT: If the arbitrage free price of this option at time n is of the form $V_n = f_n(S_n)$ for some (deterministic) function f_n, then must the functions f_n be increasing?

3. Consider infinitely many i.i.d. coin tosses where the probability of tossing heads is $p \in (0, 1/2)$ and the probability of tossing tails is $q = 1 - p$. Let $X_n = 1$ if the n^{th} coin is heads, and $X_n = -1$ otherwise. Given $S_0 \in \mathbb{N}$, and $n \in \mathbb{N}$ define $S_{n+1} = S_n + X_{n+1}$.

10 (a) Find $\alpha \in \mathbb{R}$ such that the process $e^{\alpha S_n}$ is a martingale.

10 (b) Let $M \in \mathbb{N}$, and suppose $S_0 \in \{1, 2, \dots, M-1\}$. Define $\tau = \min\{n \mid S_n \notin (0, M)\}$. Find $\boldsymbol{P}(S_\tau = M)$.
   You may leave your answer in terms of α, where α is the number you found in the previous part. Even though τ is not bounded one can show that the optional sampling theorem still applies. Feel free to assume it.