

21-370 Discrete Time Finance: Midterm 1.

2020-09-30

- This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.
- You may not, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)
- You must record yourself (audio, video and screen) and share it with me as instructed by email.
- Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.
- You have 60 minutes. The exam has a total of 4 questions and 40 points.
- Difficulty wise, $Q1 \approx Q2 \approx Q3 < Q4$. Depending on your intuition, some of you may find Q3 easier than Q1 and Q2, nevertheless, I suggest looking over the entire exam before starting. Good luck $\smile$.

- 10 1. Let $\Omega = \{-1, 1\}^{10}$ be a probability space corresponding to 10 independent coin tosses of a coin that lands heads (corresponding to +1) with probability $1/3$ and tails with probability $2/3$. Let X be a random variable such that

$$E_1 X(1, \dots, 1) = 1, \quad E_2 X(1, \dots, 1) = 1, \quad E_2 X(-1, 1, \dots, 1) = 2, \quad E_2 X(-1, -1, 1, \dots, 1) = 3.$$

Find EX .

Note: The notation $1, \dots, 1$ signifies that all omitted entries are 1. For example, $(-1, -1, 1, \dots, 1) = (-1, -1, 1, 1, 1, 1, 1, 1, 1, 1)$.

- 10 2. Let Ω be a finite probability space corresponding to N coin tosses, and $\mathcal{F}_n$ be the collection of all events that can be described using only the first n coin tosses. Let $n < N$, and suppose X and Y are two random variables such that X is $\mathcal{F}_n$ measurable. Suppose further $X + Y$ is independent of $\mathcal{F}_n$ and takes on values 1, 4, and 7 with probabilities $1/4$, $1/4$ and $1/2$ respectively. Find $E_n|X|$ and $E_n|Y|$. Your final answer may involve X and Y , but should not involve any expectations or probabilities.
- 10 3. Consider a market consisting of a money market account with interest rate 10% and a stock. The stock price is initially \$10. At every time step we flip a coin which lands heads with probability 0.3 and tails with probability 0.7, and is independent of all previous coin flips. When the coin lands heads, the stock price increases by 20%. When the coin lands tails, the stock price decreases by 20%. Consider a security that pays \$1000 at time $N = 10$ if the first 7 coin tosses are heads and the last three are tails. For any other sequence of coin tosses, the security pays nothing.
- (a) Find the arbitrage free price of this security at time 0.
- (b) Find the arbitrage free price of this security after the first coin toss.
- Your answer to both parts may depend on the outcome of the coin tosses. You should express all quantities arising in your answer as decimal numbers, correct to two decimal places. That is, don't write $2(1.23^{-10} + 3)$ if the first coin is heads in your answer. Instead evaluate this, and get a decimal number correct to two decimal places.
- 10 4. Suppose M is a martingale such that M^2 is also a martingale. Must we have $M_n = M_0$ almost surely for all $n \geq 0$? If yes, prove it. If no, find a counter example. (Note, if your solution uses fancy concepts we have not yet covered in class/homework, you would need to define / prove them to get credit.)