HW 14 Q1

$X_n \longrightarrow$ iid fair coin flips $\begin{cases} +1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$

Bet B_n at time n $\longrightarrow$ paid $B_n X_{n+1}$ at time $n+1$

(a) Cumulative gain/loss up to time n

$$= M_n = \sum_{0}^{n-1} B_k X_{k+1} = M_{n-1} + \underline{B_{n-1} X_n}$$

$$\iff M_{n+1} = \underline{M_n + B_n X_{n+1}}$$

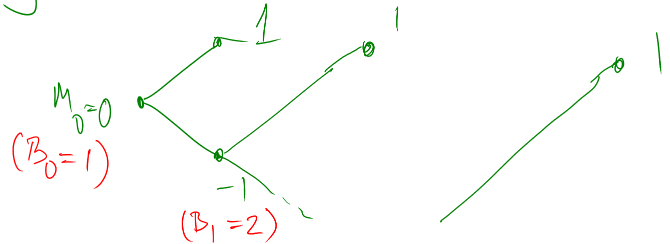
Q: Is M a mg?

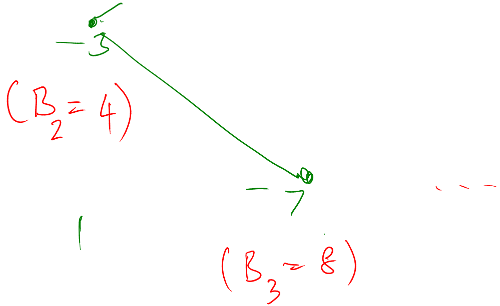
$$E_n M_{n+1} = E_n\left(M_n + B_n X_{n+1}\right) = M_n + B_n \underbrace{E X_{n+1}}_{0} = M_n$$

~~Want~~

$\Rightarrow$ M is a Mg!

(b) Double / Nothing:





τ = first time coin flips heads $(+1)$.

Q: Is τ a stopping time? Yes

$$\tau = \min\{ n \mid X_n = 1 \}$$

Q: Is $\tau < \infty$ a.s. (Yes)

Know ① $P(\tau = n) = \dfrac{1}{2^n}$

② "$P(\tau = \infty)$" $= \dfrac{1}{2^\infty}$" ← morally correct

better way to say it.

~~$\lim_{n\to\infty} P(\tau = n) \overset{?}{=} P(\tau = \infty)$~~

$$\underline{P(\tau < \infty)} = P\left(\bigcup_{n=1}^{\infty} \{\tau = n\}\right)$$

$$= \sum_{1}^{\infty} P(\tau = n) = \sum_{1}^{\infty} \frac{1}{2^n} = \underline{1}$$

$\Rightarrow \tau < \infty$ almost surely.

(c) Compute $EM_n = EM_0 = 0$ ✓ ($\because M$ is a mg)

$EM_\tau = 1$ ($\because M_\tau = 1$, see above)

(Note: OST If τ is a bdd stopping time then

$$EM_\tau = EM_0 = 0$$

But τ is not bdd so OST doesn't apply)

Also $EM_\tau^2 = 1$

(e) OST applies if (1) $E\tau < \infty$ ✓

& (2) $E_n\left(\mathbb{1}_{\{\tau \geq n\}} |M_{n+1} - M_n| \right) \leq C$

(for some C ind of n)

fails. ($M_{n+1} - M_n$ keeps growing when you lose).

Check (1): $E\tau = ?$

$$= \sum_{1}^{\infty} k\, P(\tau = k) = \underbrace{\sum_{1}^{\infty} \frac{k}{2^k}}_{} =$$

Know $\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots$

$$\Rightarrow \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

$$\Rightarrow \quad \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots$$

Put $x = \frac{1}{2}$ $\Rightarrow$ $\underbrace{\frac{1/2}{(1-1/2)^2}}_{=\,2} = \frac{1}{2} + 2\left(\frac{1}{2^2}\right) + 3\left(\frac{1}{2^3}\right) + \cdots = \sum \frac{k}{2^k}$

Q2) Coin flips iid $\begin{cases} \text{Heads with prob } p \\ \text{Tails " " } q = 1-p \end{cases}$

$a,b,c,d \in \mathbb{R}$ + $\boxed{pa + qc = 0}$

$$S^N_{n+1} = \begin{cases} S^N_n \cdot u_N & \text{if heads} \\ S^N_n \, d_N & \text{if tails.} \end{cases}$$

$$\left.\begin{aligned} u_N &= 1 + \frac{a}{\sqrt{N}} + \frac{b}{N} \\ d_N &= 1 + \frac{c}{\sqrt{N}} + \frac{d}{N} \end{aligned}\right\} \quad S_t = \lim S^N_{\lfloor Nt \rfloor}$$

Claim: $S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t \right)$

Pf: $\ln\left(\frac{S^N_{n+1}}{S^N_n}\right) = Y^N_n$. $\Rightarrow S^N_n = S_0 \exp\left(\underbrace{\sum_{k=1}^{n} Y^N_k}\right)$

Q: $\lim_{N\to\infty} \sum_{k=1}^{\lfloor Nt \rfloor} Y^N_k = ?$

Know CLT: $\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{\lfloor Nt \rfloor} X_k = \underbrace{W_t}_{\text{B.M.}}$ provided ① X_k's iid ② $EX_k = 0$ ③ $EX_k^2 = 1$

Know Y_k^N's are iid.

Let $\mu_N = E Y_k^N$ & $\sigma_N^2 = \mathrm{Var}(Y_k^N)$.

Set $X_k = \dfrac{Y_k^N - \mu_N}{\sigma_N}$ $\Rightarrow$ ① X_k's are iid.

② $E X_k = 0$ & ③ $E X_k^2 = 1$.

$$\boxed{\Rightarrow \sum_1^n Y_k = n\mu_N + \sigma_N \sum_1^n X_k}$$

Compute μ_N & σ_N.

$$Y^N_{n+1} = \ln\left(\frac{S^N_{n+1}}{S^N_n}\right)$$

① $$\mu_N = E\,Y^N_k = p\ln(u_N) + (1-p)\ln(d_N)$$

$$= p\ln\Big(1 + \underbrace{\frac{a}{\sqrt{N}} + \frac{b}{N}}_{\text{small when } N\to\infty}\Big) + q\ln\Big(1 + \underbrace{\frac{c}{\sqrt{N}} + \frac{d}{N}}_{\text{small when } N\to\infty}\Big)$$

Taylor expand $\ln(1+x) = 0 + x - \frac{x^2}{2} = x - \frac{x^2}{2} + O(x^3)$

$$\ln\left(1+\frac{a}{\sqrt{N}}+\frac{b}{N}\right) = \left(\frac{a}{\sqrt{N}}+\frac{b}{N}\right) - \frac{1}{2}\left(\frac{a}{\sqrt{N}}+\frac{b}{N}\right)^2 + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{1}{\sqrt{N}}\,a + \frac{1}{N}\left(b - \frac{a^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$\& \ \ln\left(1+\frac{c}{\sqrt{N}}+\frac{d}{N}\right) = \frac{c}{\sqrt{N}} + \frac{1}{N}\left(d - \frac{c^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right).$$

$$\Rightarrow \mu_N = p\left(\frac{a}{\sqrt{N}} + \frac{1}{N}\left(b - \frac{a^2}{2}\right)\right) + q\left(\frac{c}{\sqrt{N}} + \frac{1}{N}\left(d - \frac{c^2}{2}\right)\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{1}{\sqrt{N}}\left(pa + qc\right) + \frac{1}{N}\left(pb + qd - \frac{1}{2}\left(pa^2 + qc^2\right)\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$\underbrace{\qquad\qquad}_{0} + \frac{1}{N}\left(\alpha - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$\boxed{\text{where } \alpha = p b + q d \quad \& \quad \sigma^2 = p a^2 + q c^2}$$

$$\boxed{\text{(2) Also need to compute } \sigma_N^2 = \operatorname{Var}\left(Y_k^N\right) \overset{\text{IOU}}{=\!=} \frac{\sigma^2}{N} + O\left(\frac{1}{N^{3/2}}\right).}$$

Substitute back: $\sum_{1}^{n} Y_k^N = n\mu_N + \sigma_N \sum_{1}^{n} X_k$

$$\Rightarrow \lim_{N\to\infty} \sum_{1}^{\lfloor Nt \rfloor} Y_k^N = \lim_{N\to\infty} \left(\lfloor Nt \rfloor \left(\frac{1}{N}\left(\alpha - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right) \right) + \left(\frac{\sigma}{\sqrt{N}} + O\left(\frac{1}{N^{3/2}}\right) \right) \cdot \sum_{1}^{\lfloor Nt \rfloor} X_k \right)$$

1. $$\frac{\lfloor Nt \rfloor}{N}\left(\alpha - \frac{\sigma^2}{2}\right) \xrightarrow{N\to\infty} t\left(\alpha - \frac{\sigma^2}{2}\right)$$

2. $$\lfloor Nt \rfloor \cdot O\left(\frac{1}{N^{3/2}}\right) \xrightarrow{N\to\infty} 0$$

3. $$\frac{\sigma}{\sqrt{N}} \sum_{1}^{\lfloor Nt \rfloor} X_k \xrightarrow{N\to\infty} \sigma W_t$$

$$④\ O\left(\frac{1}{N}\right)\sum_{1}^{\lfloor Nt\rfloor} X_k \xrightarrow{N\to\infty} t\,\mathbb{E}X_k = 0 \qquad (LLN)$$

$$\Rightarrow \lim_{N\to\infty} \sum^{\lfloor Nt\rfloor} Y_k^N \longrightarrow t\left(\alpha-\frac{\sigma^2}{2}\right)+\sigma W_t$$

$$\Rightarrow S_t = \lim_{N\to\infty} S^N_{\lfloor Nt\rfloor} = S_0 \exp\left(\lim_{N\to\infty}\sum Y_k^N\right) = S_0 \exp\left(t\left(\alpha-\frac{\sigma^2}{2}\right)+\sigma W_t\right)$$

$$\alpha = pb+qd \quad \& \quad \sigma^2 = pa^2+qc^2.$$

$\mathcal{M}$

Play IOU from above. Compute $\mathrm{Var}(Y_k^N)$.

$$Y_k^N = \begin{cases} \ln u_N & \text{prob } p \\ \ln d_N & \text{prob } q. \end{cases}$$

$$\mathrm{Var}(Y_k^N) = E(Y_k^N)^2 - (E Y_k^N)^2$$

$$= p(\ln u_N)^2 + q \ln(d_N)^2 - (\mu_N)^2$$

$$= p\left(\ln\left(1 + \frac{a}{\sqrt{N}} + \frac{b}{N}\right)\right)^2 + q\left(\ln\left(1 + \frac{c}{\sqrt{N}} + \frac{d}{N}\right)\right)^2 - \left[\left(\frac{\kappa - \frac{\sigma^2}{2}}{N}\right)\right]^2 + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{1}{N}\Big(\underbrace{p a^2 + q c^2}\qquad\qquad\Big) + O\left(\frac{1}{N^{3/2}}\right)$$

$$\boxed{\begin{aligned} \ln(1+x) &\approx x + O(x^2) \\ (\ln(1+x))^2 &\approx x^2 + O(x^3) \end{aligned}}$$

$$\underline{= \frac{\Gamma^2}{N}} + O\left(\frac{1}{N^{3/2}}\right)$$

Q3) Cts time. Interest rate r.

Stock GBM

European call strike K & maturity T.

Q: R Portfolio

- Cash.
- Stock $\rightarrow$ $\boxed{\Delta(t) \text{ shares}}$

find.

Idea: ① Write as a limit of $N \rightarrow \infty$.

② Have a formula for Δ^N (in the binomial model)

③ Take limits & hope for the best!

Make it work:

① $S_t = \lim_{N \to \infty} S^N_{\lfloor Nt \rfloor}$

$S^N \to$ Binomial model $\begin{cases} r_N = r/N \\ u_N = 1 + \frac{r}{N} + \frac{u}{\sqrt{N}} \\ d_N = 1 + \frac{r}{N} - \frac{d}{\sqrt{N}} \end{cases}$

with $\sigma^2 = \tilde{p}\, u^2 + \tilde{q}\, d^2$.

& $\tilde{p} = \frac{d}{u+d}$ $\qquad \tilde{q} = \frac{u}{u+d}$

(e.g. chose $u = d = \sigma$)

② AFP $V_t = \lim_{N \to \infty} V^N_{\lfloor Nt \rfloor}$

$\Big(V_t$ = AFP in cts time mkt.

$V^N_{\lfloor Nt \rfloor}$ = AFP in Binom model.

Know for a European call $V_t = c(t, S_t)$

$c(t, x)$ = B.S. formula.

$\left(X^N_n = V^N_n \right)$.

③ X^N_n = wealth of the R. portfolio at time n in the Binom model.

Δ_n^N = # shares of stock at time n (in Binom model)

~~Know~~ $\omega = (\underbrace{\omega_1 \cdots \omega_n}_{\omega'}, \omega_{n+1}, \underline{\qquad \omega'' \qquad})$

$$\Delta_n^N(\omega) = \Delta_n^N(\omega') = \frac{V_{n+1}^N(\omega', 1) - V_{n+1}^N(\omega', -1)}{(u_N - d_N)\, S_n^N(\omega')}$$

$$\Delta_n^N = \left[\frac{C_{n+1}^N(u_N S_n^N) - C_{n+1}^N(d_N S_n^N)}{(u-d)\quad S_n^N}\right] \cdot \sqrt{N}.$$

$$= \frac{c^N_{n+1}\Big(\underbrace{\big(1+\frac{r}{N}+\frac{u}{\sqrt{N}}\big)}_{b} S^N_n\Big) - c^N_{n+1}\Big(\underbrace{\big(1+\frac{r}{N}-\frac{d}{\sqrt{N}}\big)}_{a} S^N_n\Big)}{\underbrace{\Big(\frac{u-d}{\sqrt{N}}\Big) \cdot S^N_n}_{b-a}} \Bigg|$$

$\xrightarrow{N\to\infty}$ derivative of c!

$$\Rightarrow \Delta_t = \lim_{N\to\infty} \Delta^N_{\lfloor Nt\rfloor} = \lim_{N\to\infty} \frac{c^N_{\lfloor Nt\rfloor}\Big(\big(1+\frac{r}{N}+\frac{u}{\sqrt{N}}\big) S^N_{\lfloor Nt\rfloor}\Big) - c^N_{\lfloor Nt\rfloor}\Big(\big(1+\frac{r}{N}-\frac{d}{\sqrt{N}}\big) S^N_{\lfloor Nt\rfloor}\Big)}{\Big(\frac{u-d}{\sqrt{N}}\Big) S^N_{\lfloor Nt\rfloor}}$$

$$= \lim_{N\to\infty} \frac{c\left(t, \left(1+\frac{r}{N}+\frac{u}{\sqrt{N}}\right)S_t\right) - c\left(t, \left(1+\frac{r}{N}-\frac{d}{\sqrt{N}}\right)S_t\right)}{\left(\frac{u-d}{\sqrt{N}}\right)S_t}$$

$$= \partial_x c(t, S_t)$$

//

("$\underbrace{\text{Delta Hedging Rule}}$").