8.4. **Convergence of the Binomial Model.**

(1) Let $r_N > -1$, and consider a bank that pays you interest r_N every $1/N$ time units.
(2) *Question:* Can we choose r_N so that this converges as $N \to \infty$.
(3) Let $C_0^N = 1$, $C_{n+1}^N = (1 + r_N) C_n^N$ and $C_t = \lim_{N \to \infty} C_{\lfloor Nt \rfloor}^N$.

Proposition 8.23. *If* $r \in \mathbb{R}$, $r_N = r/N$, *then* $C_t = e^{rt}$.

Remark 8.24. Note $\partial_t C_t = r C_t$. The quantity r is known as the *continuously compounded interest rate.*

Remark 8.25. If the interest rate is a constant r, then the discount factor is simply $D_t = 1/C_t = e^{-rt}$.

$\to \lfloor Nt \rfloor$

(1) Now consider the N period Binomial model, with parameters $0 < d_N < 1 + r_N < u_N$, with stock price denoted by S_n^N.
(2) Each time step for S^N denotes $1/N$ time units in real time. Can we chose u_N, d_N, r_N such that $S_t = \lim_{N \to \infty} S^N_{\lfloor Nt \rfloor}$ exists?
(3) Choose $r_N = r/N$, where $r \in \mathbb{R}$ is the continuously compounded interest rate.

Theorem 8.26. *Let $u, d > 0$ and choose*

$$u_N = 1 + \frac{r}{N} + \frac{u}{\sqrt{N}}, \qquad d_N = 1 + \frac{r}{N} - \frac{d}{\sqrt{N}}, \qquad \tilde{p} = \frac{d}{u+d}, \qquad \tilde{q} = \frac{u}{u+d}, \qquad \sigma^2 = \tilde{p}u^2 + \tilde{q}d^2 .$$

Under the risk neutral measure, the processes $S^N_{\lfloor Nt \rfloor}$ converge weakly to $S_t = S_0 e^{(r-\sigma^2/2)t + \sigma W_t}$, where W is a Brownian motion. That is, for any bounded continuous function f,

$$\lim_{N \to \infty} \tilde{\boldsymbol{E}} f(S^N_{\lfloor Nt \rfloor}) = \tilde{\boldsymbol{E}} f(S_t) = \tilde{\boldsymbol{E}} f\Big(S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)\Big)$$

Remark 8.27. S_t above is called a *Geometric Brownian motion* with mean return rate r, and volatility σ.

Remark 8.28. The fact that we took the limit under the risk neutral measure is the reason the mean return rate r is the same as the interest rate r.

Remark 8.29. In this continuous time market you have the asset (whose price is denoted by S_t), and a bank with *continuously compounded* interest rate r (i.e. discount factor is $D_t = e^{-rt}$). You can trade *continuously* in time, and we are neglecting any transaction costs.

Note : RN prob in the N period case and

$$\tilde{p}_N = \frac{1 + r_N - d_N}{u_N - d_N} = \frac{1 + \frac{r}{N} - \left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)}{\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) - \left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)}$$

$$= \frac{d/\sqrt{N}}{(u+d)/\sqrt{N}} = \frac{d}{u+d}.$$

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Pf:
$$S^N_{n+1} = \begin{cases} u_N S^N_n & \text{with prob } \tilde{p} = \frac{d}{u+d} \\ d_N S^N_n & \text{with prob } \tilde{q} = \frac{u}{u+d}. \end{cases}$$

Let $Y^N_{n+1} = \ln\left(\frac{S^N_{n+1}}{S^N_n}\right) \Rightarrow S^N_{n+1} = S^N_n \cdot e^{Y^N_{n+1}}$

$$\Rightarrow S^N_{n+1} = S^N_{n-1}\, e^{Y^N_{n+1} + Y^N_n} \quad \cdots \quad \boxed{S^N_n = S_0 \cdot e^{\sum_1^n Y_k}}$$

② Know: if $EX_n = 0$, $EX_n^2 = 1$, iid, $\frac{1}{\sqrt{N}} \sum_1^{\lfloor Nt \rfloor} X_k \longrightarrow W_t$ (B.M.)

③ Compute $\mu_N = \mathbb{E}\, Y_k^N \overset{iid}{=} \mathbb{E}\, Y_1^N$

& $\sigma_N^2 = \mathrm{Var}(Y_k^N) \overset{iid}{=} \mathrm{Var}(Y_1^N)$.

ⓐ $\mu_N := \mathbb{E}\, Y_k^N = \tilde{p} \ln u_N + \tilde{q} \ln d_N$

$$= \tilde{p} \ln\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) + \tilde{q} \ln\left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right) \quad \ldots (*)$$

ⓑ Taylor expand the fn $\ln(1+x)$:

$$\ln(1+x) = 0 + x + \frac{1}{2}(-1)x^2 + O(x^3).$$

$$\boxed{\ln(1+x) \approx x - \frac{x^2}{2} + O(x^3)}$$

(c) Use in (*):

$$\mu_N = E\, Y_k^N = \tilde{p} \ln\left(1 + \frac{r}{N} + \frac{u}{\sqrt{N}}\right) + \tilde{q} \ln\left(1 + \frac{r}{N} - \frac{d}{\sqrt{N}}\right)$$

$$= \frac{1}{\sqrt{N}}\left(\tilde{p} u - \tilde{q} d\right) + \frac{1}{N}\left(\tilde{p} r + \tilde{q} r - \frac{1}{2}\left(\tilde{p} u^2 + \tilde{q} d^2\right)\right) + O\left(\frac{1}{N^{3/2}}\right)$$

$$= \frac{1}{\sqrt{N}}\underbrace{\left(\frac{d}{u+d}\cdot u - \frac{u}{u+d}\, d\right)}_{0} + \frac{1}{N}\left(r - \frac{\sigma^2}{2}\right)$$

③ You compute $\sigma_N^2 = \mathrm{Var}\left(Y^N_{k}\right) = \dfrac{\sigma^2}{N} + O\left(\dfrac{1}{N^{3/2}}\right)$

④ Set $X^N_n = \dfrac{Y^N_n - \mu_N}{\sigma_N}$ $\iff$ $Y^N_n = \mu_N + \sigma_N X^N_n$.

Note $E X^N_n = 0$ & $\mathrm{Var}(X^N_n) = 1$.

⑤ $\Rightarrow \sum_1^n Y^N_k = \sigma_N \sum_1^n X_k + n \mu_N$

$$= \frac{\sigma}{\sqrt{N}} \sum_{1}^{n} X_k + \frac{n}{N}\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{N^{3/2}}\right)$$

(6) B.M. construction $\Rightarrow$

$$\lim_{N\to\infty} \sum_{1}^{\lfloor Nt\rfloor} Y_k = \lim_{N\to\infty}\left(\frac{\sigma}{\sqrt{N}} \sum_{1}^{\lfloor Nt\rfloor} X_k + \frac{\lfloor Nt\rfloor}{N}\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{N}}\right)\right)$$

$$= \sigma W_t + t\left(r - \frac{\sigma^2}{2}\right)$$

(7) $\Rightarrow$

$$\lim_{N\to\infty} S^N_{\lfloor Nt\rfloor} = \lim_{N\to\infty} S_0\, e^{\sum_{1}^{\lfloor Nt\rfloor} Y_k} = \underline{S_0 \exp\left(t\left(r - \frac{\sigma^2}{2}\right) + \sigma W_t\right)}$$

QED

Theorem 8.30. *Consider a security that pays $f(S_T)$ at maturity time T. The arbitrage free price of this security at time t is given by*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t\Big(D_T f(S_T)\Big) = \tilde{\boldsymbol{E}}_t\big(e^{-r(T-t)}f(S_T)\big)$$

Proof. For the Binomial model we already know $V_n^N = \frac{1}{D_n^N}\tilde{\boldsymbol{E}}_n D^N_{\lfloor NT\rfloor} f(S^N_{\lfloor NT\rfloor})$. Set $n = \lfloor Nt \rfloor$ and send $N \to \infty$. ☐

(Last time)

Proof of Theorem 8.26

Dich alone

Theorem 8.31 (Black-Scholes formula)**.** *In the above market, a European call with maturity T and strike K pays $(S_T - K)^+$ at time T. The arbitrage free price of this call at time t is $c(t, S_t)$, where*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

where $$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$ ← CDF of norml.

Proof. Let $\tau = T - t$. We know $\boxed{c(t, S(t)) = \tilde{\boldsymbol{E}}_t e^{-r\tau}(S_T - K)^+.}$ Observe first

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W_t}, \quad S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}, \quad \Longrightarrow \quad S_T = S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma(W_T - W_t)},$$

Since $W_T - W_t$ is independent of $\mathcal{F}_t$, and S_t is $\mathcal{F}_t$ measurable, by the independence lemma,

$$c(t, S_t) = \tilde{\boldsymbol{E}}_t e^{-r\tau}\big(S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma(W_T - W_t)} - K\big)^+ = \int_{\mathbb{R}} e^{-r\tau}\big(S_t e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K\big)^+ e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}.$$ ← PDF of std norml.

Now set $S_t = x$,

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \quad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-y^2/2}\,dy,$$

and observe

$$\begin{aligned} c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} x\exp\Big(\frac{-\sigma^2\tau}{2} + \sigma\sqrt{\tau}y - \frac{y^2}{2}\Big)\,dy - e^{-r\tau}KN(d_-) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} x\exp\Big(\frac{-(y-\sigma\sqrt{\tau})^2}{2}\Big)\,dy - e^{-r\tau}KN(d_-) = \boxed{xN(d_+) - e^{-r\tau}KN(d_-)}. \end{aligned}$$

□

$$W_T - W_t \sim N(0, T-t)$$

$$W_T - W_t = \sqrt{\tau} \cdot \underbrace{\left(\frac{W_T - W_t}{\sqrt{\tau}}\right)}_{N(0,1)}$$

$$\Rightarrow \frac{W_T - W_t}{\sqrt{T-t}} \sim N(0, 1)$$

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