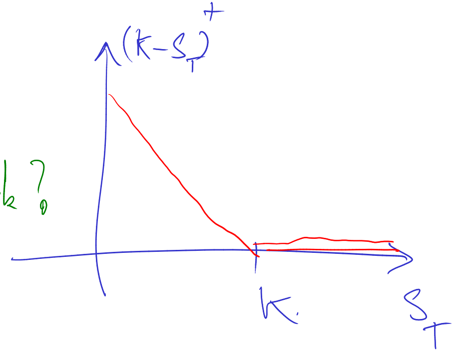
HW 13 Q 4

(a) $\bar{A}FP$ of a European Put strike K maturity T.

Payoff of a put $\longrightarrow (K - S_T)^+$ at time T.

Idea: Know B.S. formula for the price of a call.

$\longrightarrow$ Can I replicate a put using a call & Bank/stock?



Yes: Buy 1 call

Short 1 stock

$+ Ke^{-rT}$ in Bank

At maturity : $(S_T - K)^+ - S_T + K \overset{?}{=} (K - S_T)^+$ ✓

Check : ① $S_T > K$: $LHS = 0$, $RHS = 0$

② $S_T \le K$: $LHS = K - S_T$, $RHS = K - S_T$. ✓

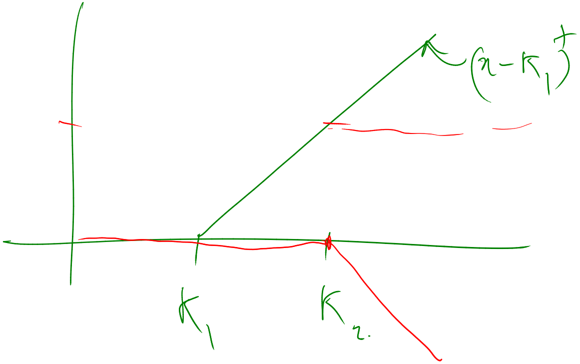
$\Rightarrow$ AFP of 1 put at time t = Wealth of the portfolio above.

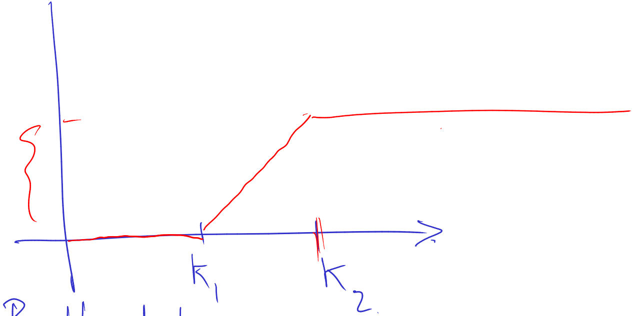
$$= \underbrace{c(t, S_t)}_{\text{given by B.S.}} - S_t + \underbrace{K e^{-rT} \cdot e^{rt}}_{K e^{-r(T-t)}}$$

(b) $0 < K_1 < K_2$.

$S_t \longrightarrow$

Option pays $\left((S_T \wedge K_2) - K_1 \right)^+$ at time T.

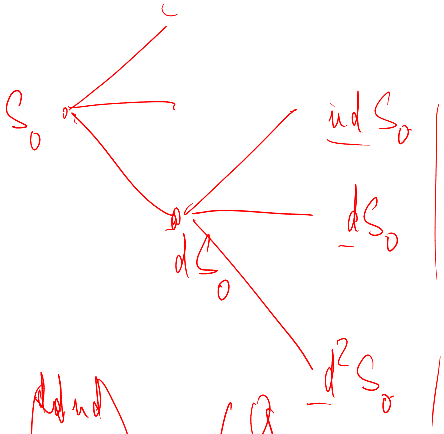


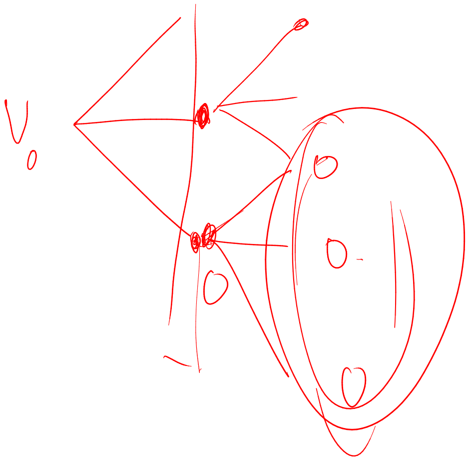


Payoff at time T

$\boxed{1c}$

Say we roll 3 on the first die. ($\omega_1 = 3$).





$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}ud\\d\\d^2\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Q5 $X_n \to$ iid. $\quad EX_n^2 = 1, \; EX_n = 0 \quad W_n^N = \frac{1}{\sqrt{N}} \sum_0^n X_k$

$f = f(x, y)$

$(n \geq sN)$

$$g_n^N(W^N_{\lfloor sN \rfloor}, W^N_n) = \tilde{E}_n f(W^N_{\lfloor sN \rfloor}, W^N_{\lfloor NT \rfloor})$$

$$u_t = \lim_{N \to \infty} g^N_{\lfloor Nt \rfloor}$$

Find a formula for u.

$$\boxed{u(t, W_s, W_t) = E_t f(W_s, W_T)}$$

Step 1 : Recursive relation for g.

$$g_n\left(W^N_{\lfloor sN\rfloor}, \underline{\underline{W^N_n}}\right) = \tilde{E}_n f\left(W^N_{\lfloor sN\rfloor}, W^N_{\lfloor NT\rfloor}\right)$$

$$= \tilde{E}_n \tilde{E}_{n+1} f\left(\qquad \right)$$

$$= \tilde{E}_n\, g_{n+1}\left(W^N_{\lfloor sN\rfloor}, W^N_{n+1}\right)$$

$$= \tilde{E}_n\, g_{n+1}\left(W^N_{\lfloor sN\rfloor}, W^N_n + \frac{1}{\sqrt{N}} X_{n+1}\right)$$

$$\boxed{\begin{aligned} x &= W^N_{\lfloor sN\rfloor} \\ y &= W^N_n \end{aligned}} \longrightarrow = \sum_{i=1}^{M} g_{n+1}\left(W^N_{\lfloor sN\rfloor}, W^N_n + \frac{1}{\sqrt{N}} x_i\right) p_i$$

$$\Rightarrow g_n(x,y) = \sum_{i=1}^{M} g_{n+1}\left(x,\ y + \frac{1}{\sqrt{N}} x_i\right) p_i$$

$$(\because \mathrm{Var}\, X_n = 1).$$

$$= \sum_i p_i \Big[g_{n+1}(x,y) + \underbrace{\partial_y g_{n+1}(x,y) \frac{x_i}{\sqrt{N}}}_{=0\ (\because \sum x_i p_i = 0)} + \overbrace{\frac{1}{2} \partial_y^2 g_{n+1}(x,y) \frac{x_i^2}{N}}^{\frac{1}{2N} \partial_y^2 g_{n+1}(x,y) \cdot 1} + O\left(\frac{1}{N^{3/2}}\right) \Big]$$

$$= g_{n+1}(x,y) + 0 + \frac{1}{2N} \partial_y^2 g(x,y)$$

$$\Rightarrow \underbrace{\frac{g_n(x,y) - g_{n+1}(x,y)}{1/n}}_{\downarrow \; -\partial_t u(x,y)} = \underbrace{\frac{1}{2}\partial_y^2 g_{n+1}(x,y)}_{\downarrow \; \frac{1}{2}\partial_y^2 u(x,y)} + O\left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow \partial_t u + \frac{1}{2}\partial_y^2 u = 0$$ $\longrightarrow$ Use the sol formula for u from <u>class</u>

$$u(t,x,y) = \int_{z\in\mathbb{R}} f(x,z)\, \boxed{\overbrace{F(t,x,y-z)}^{G_{T-t}(x,y-z)}\, dz}$$

$\downarrow$

PDF of

$$E f(Z) = \int f(z) \cdot \widetilde{p_Z(z)}\, dz \qquad p_Z = \text{PDF of } Z.$$

Should (hopefully) get $u(t, W_s, W_t) = E_t(f(W_s, W_T))$

$$= \int_{\mathbb{R}} f(W_s, -) \;\_\_\_\_\_\_$$